King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Math 432 - Final Exam

AY 2023-2024 (Term 232)

Time Allowed: 180 Minutes

Name: .... **ID number:** ....

- Textbook, notes, mobiles and smart devices are not allowed in this exam.
- Write neatly and legibly. You may lose points for messy work.
- Show all your work. No points for answers without justification.

Question	**Marks**	**Max Marks**
1		40
2		10
3		15
4		15
5		15
6		10
7		15
8		20
Total		140

Question 1

Suppose A is symmetric positive definite and Q is an orthogonal matrix. Answer **True** or **False**, give a reason if true or a counterexample if false:

1. $Q^T AQ$ is a diagonal matrix. (__________)

2. $Q^T AQ$ is symmetric positive definite. (__________)

3. $Q^T AQ$ has the same eigenvalues as A. (__________)

4. e^{-A} is symmetric positive definite. (__________)

5. Every positive definite matrix is invertible. (__________)

6. The only positive definite projection matrix is $P = I$. (__________)

7. Any diagonal matrix with positive diagonal entries is positive definite. (__________)

8. Any symmetric matrix with positive determinant is positive definite. (__________)

9. Every invertible matrix can be diagonalized. (__________)

10. Every diagonalizable matrix can be inverted. (__________)

Question 2

For which s and t are A and B positive definite?

$$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{bmatrix}.$$

Question 3

For the given matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

1. Find the singular value decomposition (SVD) of the matrix A, i.e., $A = U\Sigma V^T$.
2. Define the pseudoinverse and find A^+.
3. Find the norm and condition number of A.
4. Which four vectors give orthonormal bases for the four fundamental subspaces of A?

Question 4

Find the minimum-length least-squares solution $x^+ = A^+ b$ to the following:

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

You can compute A^+, or find the general solution to $A^T A\tilde{x} = A^T b$ and choose the solution that is in the row space of A.

Verify that $x^+ = A^+ b$ is in the row space and $A^T A x^+ = A^T b$.

Question 5

1. Define the Rayleigh quotient $R(x)$ and state the Rayleigh's Principle.
2. Show that the diagonal entries of any symmetric matrix are between λ_{min} and λ_{max}.
3. For any symmetric matrix A, compute the ratio $R(x)$ for the special choice $x = (1, \ldots, 1)$. How is the sum of all entries a_{ij} related to λ_{min} and λ_{max}?

Question 6

Use Gram–Schmidt to construct an orthonormal pair q_1, q_2 from $a_1 = (4, 5, 2, 2)^T$ and $a_2 = (1, 2, 0, 0)^T$. Express a_1 and a_2 as combinations of q_1 and q_2, and find the triangular R in $A = QR$.
(A is 4×2 matrix with columns a_1, a_2)

Question 7

A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \quad \text{and} \quad \frac{dw}{dt} = v - w.$$

1. Show that the total $v + w$ is constant (40 people).
2. Find the matrix A in $\begin{bmatrix} \frac{dv}{dt} \\ \frac{dw}{dt} \end{bmatrix} = A \begin{bmatrix} v \\ w \end{bmatrix}$, and its eigenvalues and eigenvectors.
3. What are v and w at $t = 1$?

Question 8

Suppose the matrices in $PA = LU$ are

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 3 & 2 \\ 2 & -1 & 4 & 2 & 1 \\ 4 & -2 & 9 & 1 & 4 \\ 2 & -1 & 5 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 4 & 2 & 1 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. What is the rank of A?
2. What is a basis for the row space of A?
3. *True or false*: The rows $1, 2, 3$ of A are linearly independent.
4. What is a basis for the column space of A?
5. What is the dimension of the left nullspace of A?