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College of Computing and Mathematics
Department of Mathematics

MATH435 – Ordinary Differential Equations
EXAM 1

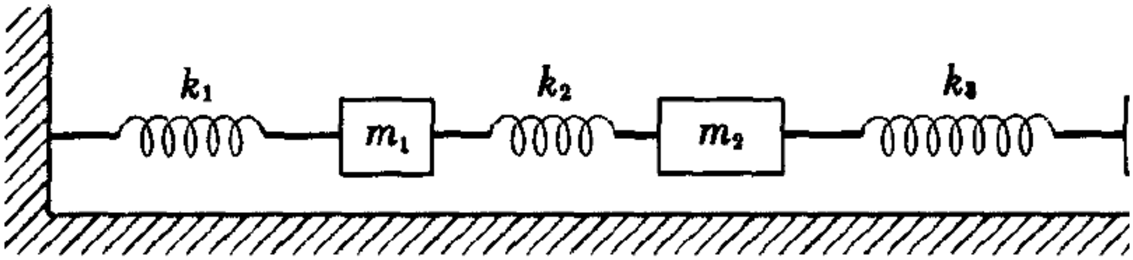
AY 2021-2022 **(Term 212)**

Time allowed: **90** Minutes

Question #	Mark	Max Mark
1		20
2		20
3		20
4		20
5		20
Total		**100**

Question 1

a) Write a system of differential equations (equation of motion) for the following mass-spring system considering the air resistance.



b) Write a system of first-order differential equations equivalent to the system in **a)** and write the vector-matrix notation of the first-order system.

Solution :

a) $$m_1 \frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1) - a\frac{dx_1}{dt} + b\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right)$$

$$m_2 \frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1) - k_3x_2 - b\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) - c\frac{dx_2}{dt}$$

b) put $y_1 = x_1$, $y_2 = \frac{dx_1}{dt}$, $y_3 = x_2$, $y_4 = \frac{dx_2}{dt}$

$$y_1' = y_2$$
$$y_2' = \frac{1}{m_1}\left[-k_1y_1 + k_2(y_3 - y_1) - ay_2 + b(y_4 - y_2)\right]$$
$$y_3' = y_4$$
$$y_4' = -\frac{1}{m_2}\left[k_2(y_3 - y_1) + k_3y_3 + b(y_4 - y_2) + cy_4\right]$$

Vector-matrix form:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1-k_2}{m_1} & \frac{-a-b}{m_1} & \frac{k_2}{m_1} & \frac{b}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{1}{m_2} & \frac{-b}{m_2} & \frac{-k_2-k_3}{m_2} & \frac{-b-c}{m_2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$$\Rightarrow \quad y' = Ay$$

Question 2

a) Discuss the problem of existence and uniqueness of solution of the initial value problem for the system.

$$\begin{cases} y_1' = y_2 + \dfrac{y_3}{t-1} \\ y_2' = y_1 \sin t + t^2 y_3 \\ y_3' = y_1 - y_2^2 + y_3 \end{cases}, \qquad \boldsymbol{y}(t_0) = \boldsymbol{\eta}$$

b) Discuss the problem of existence and uniqueness of solution of the following initial value problem. Solve the problem and give the interval of definition of the solution.

$$\frac{dy}{dt} = -\frac{y}{t+1}, \qquad y(0) = 1$$

Solution:

a)

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad f(t,y) = \begin{pmatrix} y_2 + \dfrac{y_3}{t-1} \\ y_1 \sin t + t^2 y_3 \\ y_1 - y_2^2 + y_3 \end{pmatrix}$$

$$\frac{\partial f}{\partial y_1} = \begin{pmatrix} 0 \\ \sin t \\ 1 \end{pmatrix}, \quad \frac{\partial f}{\partial y_2} = \begin{pmatrix} 1 \\ 0 \\ -2y_2 \end{pmatrix}, \quad \frac{\partial f}{\partial y_3} = \begin{pmatrix} \dfrac{1}{t-1} \\ t^2 \\ 1 \end{pmatrix}$$

$f, \dfrac{\partial f}{\partial y_1}, \dfrac{\partial f}{\partial y_2}, \dfrac{\partial f}{\partial y_3}$ are all continuous for $t \neq 1$

Thus, the IVP has a unique solution if $t_0 \neq 1$

b) $f(t,y) = -\dfrac{y}{t+1}$, $\quad \dfrac{\partial f}{\partial y} = -\dfrac{1}{t+1}$

Both f, $\dfrac{\partial f}{\partial y}$ are continuous for $t \neq -1$.

Thus, the IVP has a unique solution.

$$\frac{dy}{dt} = -\frac{y}{t+1}$$

$$\Rightarrow \quad \frac{dy}{y} = -\frac{dt}{t+1}$$

$$\Rightarrow \quad \ln|y| = -\ln|t+1| + \ln c$$

$$\Rightarrow \quad y = \frac{c}{t+1}$$

$y(0)=1 \quad \Rightarrow \quad 1 = \dfrac{c}{1} \quad \Rightarrow \quad c = 1$

and $\quad I = (-1, \infty)$

containing $t=0$.

Question 3 Consider the differential equation

$$y' = g(t)y$$

Assume that

$$\int_0^t |g(s)| ds \le t, \qquad t \ge 0,$$

$$\int_t^0 |g(s)| ds \le -t, \qquad t \le 0.$$

Show that $|y(t)| \le |y(0)| \exp(|t|)$.

Solution :

Case 1 : $t \geq 0$, integrating both sides of the DE from 0 to t yields

$$y(t) = y(0) + \int_0^t g(s)\, y(s)\, ds, \qquad t \geq 0$$

$$\Rightarrow \quad |y(t)| \le |y(0)| + \int_0^t |g(s)|\, |y(s)|\, ds$$

By the Gronwall's inequality

$$|y(t)| \le |y_0| \exp\left(\int_0^t g(s)\, ds\right)$$

$$\le |y_0| e^t, \qquad t \geq 0$$

Case 2: $t \leq 0$, integrating both sides of the DE from t to 0 yields

$$y(0) - y(t) = \int_t^0 g(s)\, y(s)\, ds, \qquad t \leq 0$$

$$\Rightarrow \quad y(t) = y(0) - \int_t^0 g(s)\, y(s)\, ds,$$

$$\Rightarrow \quad |y(t)| \leq |y(0)| + \int_t^0 |g(s)|\, |y(s)|\, ds$$

By Gronwall's inequality:

$$y(t) \leq |y(0)| \exp\left(\int_t^0 |g(s)|\, ds \right)$$
$$\leq |y(0)|\, e^{-t}, \qquad t \leq 0$$

Hence,

$$|y(t)| \leq |y(0)|\, e^{|t|}, \qquad -\infty < t < \infty$$

Question 4 Let

$$\boldsymbol{A}(t) = \begin{pmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{pmatrix}, \qquad \boldsymbol{f}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$$

a) Show that

$$\boldsymbol{\Phi}(t) = \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix}$$

is a fundamental matrix for the system $\boldsymbol{y}' = \boldsymbol{A}(t)\boldsymbol{y}$ on any interval I not including the origin.

b) Consider the system $y' = \boldsymbol{A}(t)y + \boldsymbol{f}(t)$. Find the solution $\boldsymbol{\phi}$ the nonhomogeneous system satisfying the initial condition $\boldsymbol{\phi}(1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Solution:

a) $$\Phi'(t) = \begin{pmatrix} 2t & 1 \\ 2 & 0 \end{pmatrix}$$

$$A(t)\,\Phi(t) = \begin{pmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{pmatrix}\begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2t & 1 \\ 2 & 0 \end{pmatrix}$$

$$\Phi'(t) = A(t)\,\Phi(t)$$

and $$\det \Phi(t) = t^2 - 2t^2$$
$$= -t^2 \neq 0 \quad \text{for } t \neq 0$$

So, Φ is fundamental matrix on an interval I not containing the origin.

b) $$\psi(t) = \Phi(t)\int_1^t \Phi^{-1}(s)\, f(s)\, ds$$

$$= \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix} \int_1^t -\frac{1}{s^2}\begin{pmatrix} 1 & -s \\ -2s & s^2 \end{pmatrix}\begin{pmatrix} s \\ 1 \end{pmatrix} ds$$

$$= \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix} \int_1^t -\frac{1}{s^2}\begin{pmatrix} 0 \\ -s^2 \end{pmatrix} ds$$

$$= \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix} \int_1^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} ds$$

$$= \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix}\begin{pmatrix} 0 \\ t-1 \end{pmatrix} = \begin{pmatrix} t^2-t \\ t-1 \end{pmatrix}$$

$$\phi(t) = \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} t^2-t \\ t-1 \end{pmatrix}$$

$$\phi(1) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{aligned} c_1 + c_2 &= 1 \\ 2c_1 + c_2 &= -1 \\ \hline -c_1 \quad &= 2 \end{aligned} \;-\; \Longrightarrow c_1 = -2,\; c_2 = 3$$

$$\phi(t) = \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} t^2-t \\ t-1 \end{pmatrix} = \begin{pmatrix} -t^2+2t \\ -3t+2 \end{pmatrix}$$

Question 5 Find the fundamental matrix e^{At} of the system

$$\boldsymbol{Y}' = \boldsymbol{AY}, \qquad \boldsymbol{A} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}$$

Solution: Eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1$ with multiplicities $n_1 = 1$, $n_2 = 2$

$$(A - \lambda_1 I)V_1 = (A + I)V_1 = 0$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} 2k_1 + 2k_2 + 2k_3 &= 0 \\ 3k_1 + 2k_3 &= 0 \end{aligned}$$

$$\Rightarrow \begin{aligned} 2k_1 + 2k_2 - 3k_1 &= 0 \\ 2k_3 &= -3k_1 \end{aligned}$$

$$\Rightarrow \begin{aligned} 2k_2 &= k_1 \\ 2k_3 &= -3k_1 \end{aligned} \Rightarrow V_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} c_1$$

$$(A - \lambda_2 I)^2 = (A - I)^2 = \begin{pmatrix} -2 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \\ -6 & 0 & 0 \end{pmatrix}$$

$$(A - I)^2 V_2 = 0$$

$$\Rightarrow \begin{pmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \\ -6 & 0 & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow k_1 = 0$$

$$V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} c_2$$

$$\eta = v_1 + v_2 \Rightarrow \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} c_1 + \begin{pmatrix} 0 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\eta_1 = 2c_1 \Rightarrow c_1 = \frac{\eta_1}{2}$$

$$\eta_2 = c_1 + c_2 \Rightarrow c_2 = \eta_2 - \frac{\eta_1}{2}$$

$$\eta_3 = -3c_1 + c_3 \Rightarrow c_3 = \eta_3 + \frac{3}{2}\eta_1$$

$$v_1 = \begin{pmatrix} \eta_1 \\ \frac{1}{2}\eta_1 \\ -\frac{3}{2}\eta_1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ \eta_2 - \frac{1}{2}\eta_1 \\ \eta_3 + \frac{3}{2}\eta_1 \end{pmatrix}$$

$$e^{tA}\eta = e^{-t}v_1 + e^{t}\left(I + (A-I)t\right)v_2$$

$$= e^{-t}\begin{pmatrix} 1 \\ 1/2 \\ -3/2 \end{pmatrix}\eta_1 + e^{t}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & 0 & 0 \end{pmatrix} t\right]\begin{pmatrix} 0 \\ \eta_2 - \frac{1}{2}\eta_1 \\ \eta_3 + \frac{3}{2}\eta_1 \end{pmatrix}$$

$$= e^{-t}\eta_1\begin{pmatrix} 1 \\ 1/2 \\ -3/2 \end{pmatrix} + e^{t}\begin{pmatrix} 1-2t & 0 & 0 \\ 2t & 1 & 2t \\ 3t & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \eta_2 - \frac{1}{2}\eta_1 \\ \eta_3 + \frac{3}{2}\eta_1 \end{pmatrix}$$

$$= e^{-t}\eta_1\begin{pmatrix} 1 \\ 1/2 \\ -3/2 \end{pmatrix} + e^{t}\begin{pmatrix} 0 \\ \eta_2 - \frac{1}{2}\eta_1 + \left(\eta_3 + \frac{3}{2}\eta_1\right)(2t) \\ \eta_3 + \frac{3}{2}\eta_1 \end{pmatrix}$$

$$\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \quad \phi_1(t) = \begin{pmatrix} e^{-t} \\ \frac{1}{2}\left(e^{-t} - e^{t}\right) + 3te^{t} \\ \frac{3}{2}\left(e^{t} - e^{-t}\right) \end{pmatrix}$$

$$\eta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \phi_2(t) = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$$

$$\eta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \phi_3(t) = \begin{pmatrix} 0 \\ 2te^t \\ e^t \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{-t} & 0 & 0 \\ \frac{1}{2}(e^{-t} - e^t) + 3te^t & e^t & 2te^t \\ \frac{3}{2}(e^t - e^{-t}) & 0 & e^t \end{pmatrix} = \Phi(t)$$

with $\Phi(0) = I$