جامعة الملك فهد للبترول والمعادن
King Fahd University of Petroleum & Minerals

College of Computing and Mathematics
Department of Mathematics

MATH435 – Ordinary Differential Equations
EXAM 2

AY 2021-2022 **(Term 212)**

Time allowed: **120** Minutes

Solution

Question #	Mark	Max Mark
1		20
2		20
3		15
4		25
5		20
Total		**100**

Question 1 Let

$$\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}.$$

a) [9pts] Determine the canonical form $\mathbf{J}$ of $\mathbf{A}$ and find the corresponding matrix $\mathbf{T}$ such that $\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$

b) [3pts] Draw the phase portrait of the canonical system $\mathbf{z}' = \mathbf{J}\mathbf{z}$

c) [8pts] Find a fundamental matrix of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ using the change of variables $\mathbf{y} = \mathbf{T}\mathbf{z}$.

Solution:

a) $$\begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(2-\lambda)+1=0 \Rightarrow \lambda^2-6\lambda+9=0$$

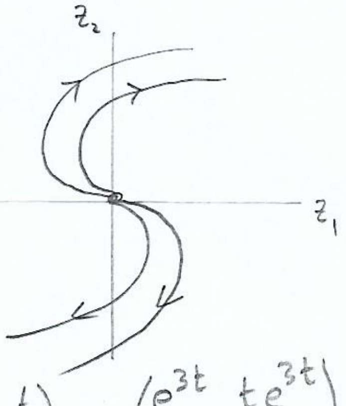
$$\lambda_1 = \lambda_2 = 3$$

$$(A-3I)e_1 = 0 \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow k_1 = k_2 \Rightarrow e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(A-3I)e_2 = e_1 \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow k_1 = k_2+1 \Rightarrow e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} T &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ T^{-1} &= \frac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned}\right] \Rightarrow J = T^{-1}AT = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

b) $$z' = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} z \Rightarrow \text{phase portrait}$$



c) $$J = B + C, \text{ where } B = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

B and C are Commute, $C^2 = 0$

$$e^{Jt} = e^{(B+C)t} = e^{Bt}e^{Ct} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{3t} & te^{3t} \\ 0 & e^{3t} \end{pmatrix}$$

A fundamental matrix of $y' = Ay$ is

$$\Phi(t) = e^{At} = Te^{Jt}T^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} e^{3t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= e^{3t}\begin{pmatrix} 1 & t+1 \\ 1 & t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = e^{3t}\begin{pmatrix} t+1 & -t \\ t & 1-t \end{pmatrix}$$

Question 2

Consider the non-homogeneous linear system

$$X' = AX + \begin{pmatrix} e^{-t} \\ 2e^{-2t} \end{pmatrix}$$

where A is a constant matrix of real eigenvalues λ_1 and λ_2. In each of the following cases, determine whether $\lim_{t\to\infty}|X(t)| = 0$, or $\lim_{t\to\infty}|X(t)| = \infty$, or $|X(t)| \le K$ for all $t \ge 0$.

a) $\lambda_1 = 0, \lambda_2 < 0$ [10pts]

b) $\lambda_1 \le -\frac{1}{2}, \lambda_2 \le -\frac{1}{2}$ [10pts]

Solution:

Let $F(t) = \begin{pmatrix} e^{-t} \\ 2e^{-2t} \end{pmatrix}$

a) $X' = AX + F(t)$

$$\left.\begin{array}{l} (A - \lambda_1 I) e_1 = 0 \\ (A - \lambda_2 I) e_2 = 0 \end{array}\right] \Rightarrow T = \begin{pmatrix} e_1 & e_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$$

Let $X = TY$

$$\Rightarrow TY' = ATY + F(t)$$

$$Y' = T^{-1}ATY + T^{-1}F(t)$$

$$= JY + T^{-1}F(t)$$

$$Y(t) = e^{Jt}Y(0) + e^{Jt}\int_0^t e^{-Js} T^{-1}F(s)\,ds$$

$$e^{Jt} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

$$\lambda_1 = 0, \quad \lambda_2 < 0 \quad \Rightarrow e^{Jt} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

$$|e^{Jt}| \le 1 + e^{\lambda_2 t} \le 2, \quad \forall t \ge 0$$

$$|Y(t)| \le |Y(0)|\,|e^{Jt}| + \int_0^t e^{J(t-s)} |T^{-1}|\,|F(s)|\,ds$$

$$\le 2|Y(0)| + 2|T^{-1}| \int_0^t (e^{-s} + e^{-2s})\,ds$$

$$= 2|Y(0)| + 2|T^{-1}| \left(-e^{-t} - \tfrac{1}{2}e^{-2t} + 2\right)$$

$$\le 2|Y(0) + 4|T^{-1}| \le M$$

$$X(t) = T\,Y(t) \quad \Rightarrow \quad |X(t)| \leq K, \quad \forall\, t \geq 0$$

b) $\lambda_1 \leq -\frac{1}{2}$, $\lambda_2 \leq -\frac{1}{2}$, $\quad -\frac{1}{2} \geq \max\{\lambda_1, \lambda_2\}$

$$|e^{Jt}| \leq e^{\lambda_1 t} + e^{\lambda_2 t} \leq 2e^{-\frac{1}{2}t}, \quad t \geq 0$$

$$|Y(t)| \leq 2\,|Y(0)|\,e^{-\frac{1}{2}t} + 2\int_0^t e^{-\frac{1}{2}(t-s)}\,|T^{-1}|\,(e^{-s} + e^{-2s})\,ds$$

$$\leq 2\,|Y(0)|\,e^{-\frac{1}{2}t} + 2\,|T^{-1}|\,e^{-\frac{1}{2}t}\int_0^t \left(e^{-\frac{1}{2}s} + e^{-\frac{3}{2}s}\right) ds$$

$$\leq 2\,|Y(0)|\,e^{-\frac{1}{2}t} + 2\,|T^{-1}|\,e^{-\frac{1}{2}t}\left(-2e^{-\frac{1}{2}t} - \frac{2}{3}e^{-\frac{3}{2}t} + 2 + \frac{2}{3}\right)$$

$$\leq \left[2\,|Y(0)| + \frac{16}{3}\,|T^{-1}|\right] e^{-\frac{1}{2}t}, \quad t \geq 0$$

$$\lim_{t\to\infty} |Y(t)| = 0 \quad \Longrightarrow \quad \lim_{t\to 0} |X(t)| = 0$$

Question 3 Consider the IVP

$$y' = f(x,y), \qquad y(-3) = 2$$

where

$$f(x,y) = \frac{e^{y^2-1}}{3 - x^2y^2}$$

Let

$$D = \{(x,y) \in \mathbb{R}^2, |x+3| \le 1, |y-2| \le 1\}$$

a) [8pts] Find two constants $M > 0$ and $K > 0$ such that

$$|f(x,y)| \le M, \qquad \left|\frac{\partial f}{\partial y}(x,y)\right| \le K, \qquad \text{for all } (x,y) \in D.$$

b) [7pts] Show that the IVP has a unique solution $y(x),\ x \in I$ (give explicitly the interval I)

Solution

a) $-4 \le x \le -2, \quad 1 \le y \le 3 \Rightarrow 1 \le y^2 \le 9 \Rightarrow 0 \le y^2 \le 8$

$\Rightarrow 1 \le e^{y^2-1} \le e^8$

$4 \le x^2 \le 16 \Rightarrow 4 \le x^2y^2 \le 144 \Rightarrow -141 \le 3 - x^2y^2 \le -1$

$\Rightarrow |3 - x^2y^2| \ge 1$

$|f(x,y)| = \left|\frac{e^{y^2-1}}{3-x^2y^2}\right| \le e^8, \quad (x,y) \in D$

$M = e^8.$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2y\,e^{y^2-1}}{3-x^2y^2} + \frac{2x^2y\,e^{y^2-1}}{(3-x^2y^2)^2}$$

$2 \le 2y \le 6, \quad 8 \le 2x^2y \le 96$

$$\Rightarrow \left|\frac{\partial f}{\partial y}(x,y)\right| \le |2y|\left|\frac{e^{y^2-1}}{3-x^2y^2}\right| + |2x^2y|\left|\frac{e^{y^2-1}}{(3-x^2y^2)^2}\right|$$

$$\le 6e^8 + 96e^8 = 102\,e^8 = K$$

b) $\alpha = \min\{1, 1/e^8\} = e^{-8}$, $f, \frac{\partial f}{\partial y}$ are continuous in D

$\Rightarrow$ the IVP has a unique solution $y(x)$,

$x \in I = [-3 - e^{-8}, -3 + e^{-8}]$

Question 4 Consider the linear system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \mathbf{A}(t)\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \mathbf{A}(t) = \begin{pmatrix} \dfrac{\sin t}{1-\cos t} - 2 & 0 \\ -1 & 1 \end{pmatrix}$$

a) [2pts] Find the smallest positive number ω such that $\mathbf{A}(t+\omega) = \mathbf{A}(t)$, for all $t \geq 0$.
b) [10pts] Solve the system and give the general solution
c) [10pts] Find the characteristic multipliers and characteristic exponents of the system.
d) [3pts] Does the system have a periodic solution? (Justify your answer)

Solution:

a) $A(t+2\pi) = A(t)$, $\qquad \omega = 2\pi$

b) $x' = \left(\dfrac{\sin t}{1-\cos t} - 2\right) x$, $\qquad y' = y - x$

$$\frac{x'}{x} = \frac{\sin t}{1-\cos t} - 2 \Rightarrow \ln|x| = \ln|1-\cos t| - 2t + c$$

$$x = (1-\cos t)\, e^{-2t} c_1$$

$$y' = y - (1-\cos t)\, e^{-2t} c_1$$

$$y' - y = e^{-2t}(\cos t - 1)\, c_1$$

$$e^{-t}(y'-y) = e^{-3t}(\cos t - 1)\, c_1$$

$$e^{-t} y = c_1\left[\int e^{-3t}(\cos t - 1)\right]$$

$$= c_1\left[\frac{1}{30} e^{-3t}(10 - 9\cos t + 3\sin t) + c\right]$$

$$\Rightarrow \quad y = \frac{c_1}{30} e^{-2t}(10 - 9\cos t + 3\sin t) + c_2 e^{t}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1-\cos t \\ \frac{1}{30}(10 - 9\cos t + 3\sin t) \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}$$

$$\phi_1(t) = \begin{pmatrix} 1-\cos t \\ \frac{1}{30}(10 - 9\cos t + 3\sin t) \end{pmatrix} e^{-2t}, \qquad \phi_2(t) = \begin{pmatrix} 0 \\ e^{t} \end{pmatrix}$$

$$\phi_1(t+2\pi) = \phi_1(t)\, e^{-4\pi} \Rightarrow \lambda_1 = e^{-4\pi} \Rightarrow \rho_1 = \frac{1}{2\pi}(-4\pi) = -2$$

$$\phi_2(t+2\pi) = \phi_2(t)\, e^{2\pi} \Rightarrow \lambda_2 = e^{2\pi} \Rightarrow \rho_2 = \frac{1}{2\pi}(2\pi) = 1$$

c) There is no periodic solution,
since $\lambda = 1$ (or $\lambda = -1$) is not a multiplier.

Question 5 Consider the first order IVP

$$\frac{dy}{dt} = f(t, y), \quad y(0) = y_0$$

Let $D = \{(t, y) \in \mathbb{R}^2, |t| \le \alpha, |y - y_0| \le \beta\}$, $\alpha, \beta > 0$. Assume that f is continuous in D and

$$|f(t, y) \le K, \quad \text{for all } (t, y) \in D$$

$$|f(t, x) - f(t, y)| \le L|x - y|, \qquad \text{for all } (t, x), (t, y) \in D.$$

Define $I = \{t \in \mathbb{R}, |t| \le h\}$, $h = \min\left(\alpha, \frac{\beta}{K}\right)$, and the sequence $\{y_n\}$ by

$$y_n(t) = y_0 + \int_0^t f(s, y_{n-1}(s))ds, \qquad n = 1,2,3, \dots, \qquad t \in I$$

a) [13pts] Show that $y_n(t)$ converges to a function $y(t)$, $t \in I$.
b) [7pts] Assuming that $y(t)$ is continuous on I, prove that $y(t)$ satisfies the integral equation

$$y(t) = y_0 + \int_0^t f(s, y(s))ds, \qquad t \in I$$

Hint: you may use the following result:

For $j = 0,1,2, \dots, \quad t \in I$,

$$|y_{j+1}(t) - y_j(t)| \le \frac{KL^j|t|^{j+1}}{(j+1)!}$$

a) $$y_n(t) = y_0(t) + \sum_{j=0}^{n} (y_{j+1}(t) - y_j(t))$$

$$|y_n(t)| \le |y_0(t)| + \sum_{j=0}^{n} |y_{j+1}(t) - y_j(t)|$$

But $$|y_{j+1}(t) - y_j(t)| \le \frac{KL^j|t|^{j+1}}{(j+1)!} \le \frac{KL^j h^{j+1}}{(j+1)!} = \frac{K}{L}\frac{(Lh)^{j+1}}{(j+1)!}$$

We know that

$$\sum_{j=0}^{\infty} \frac{(Lh)^{j+1}}{(j+1)!} \quad \text{converges to } 1 + e^{Lh}$$

Hence

$$y_0(t) + \sum_{j=0}^{\infty} [y_{j+1}(t) - y_j(t)] \quad \text{converges}$$

uniformly and absolutely on I

We set

$$y(t) = y_0(t) + \sum_{j=0}^{\infty} \left(y_{j+1}(t) - y_j(t) \right), \quad t \in I$$

and $\lim_{n \to \infty} y_n(t) = y(t), \quad t \in I$

b) $y(t) - y(0) - \int_0^t f(s, y(s))\, ds$

$$= y(t) - y_n(t) + \int_0^t f(s, x_{n-1}(s))\, ds - \int_0^t f(s, x(s))\, ds$$

$$\left| y(t) - y(0) - \int_0^t f(s, y(s))\, ds \right|$$

$$\leq |y(t) - y_n(t)| + \int_0^t |f(s, y_{n-1}(s)) - f(s, y(s))|\, ds$$

$$\leq |y(t) - y_n(t)| + Lh \max_{-h \leq s \leq h} |y(s) - y_{n-1}(s)|$$

$$\longrightarrow 0 \quad \text{if} \quad n \to \infty$$

Hence

$$y(t) = y(0) + \int_0^t f(s, y(s))\, ds$$