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College of Computing and Mathematics
Department of Mathematics

MATH435 – Ordinary Differential Equations
FINAL EXAM

AY 2021-2022 **(Term 212)**

Time allowed: **150** Minutes

Name : Solution

STUID# :

Question #	Mark	Max Mark
1		15
2		12
3		20
4		10
5		14
6		15
7		14
Total		**100**

Problem 1 Discuss the existence and uniqueness of the solution to the following initial value problem.

a) $y'' + k_1 y' + k_2 \sin y = 0, \;\; y(a) = b, y'(a) = c.$

b) $y' = y^{1/3} + t, \;\; y(\alpha) = \beta.$

c) $\begin{cases} y_1' = ty_1 + y_2 + \cos t \\ y_2' = (\sin t)y_1 + y_2 + \frac{e^t}{t} \\ y_1(1) = 1, \;\; y_2(1) = 0. \end{cases}$

Solution:

a) put $y_1 = y$ and $y_2 = y'$. We have

$$\begin{cases} y_1' = y_2 \\ y_2' = -k_1 y_2 - k_2 \sin y_1 \end{cases}$$

Let $F(t,y) = \begin{pmatrix} y_2 \\ -k_1 y_2 - k_2 \sin y_1 \end{pmatrix}$

$$F_{y_1} = \begin{pmatrix} 0 \\ -k_2 \cos y_1 \end{pmatrix}, \quad F_{y_2} = \begin{pmatrix} 1 \\ -k_1 \end{pmatrix}$$

F, F_{y_1}, F_{y_2} are all continuous everywhere. Thus, the IVP has a unique solution for any choice of initial conditions.

b) Let $F(t,y) = y^{1/3} + t$. $F_y = \frac{1}{3} y^{-2/3}$.

F_y is not continuous when $y = 0$.
the IVP has a unique solution if $\beta \neq 0$.

c) Let $F(t,y) = \begin{pmatrix} ty_1 + y_2 + \cos t \\ (\sin t) y_1 + y_2 + \frac{e^t}{t} \end{pmatrix}$, $F_{y_1} = \begin{pmatrix} t \\ \sin t \end{pmatrix}$

$$F_{y_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since F, F_{y_1}, F_{y_2} are continuous for $t=1, y_1 = 1, y_2 = 0$,

The IVP has a unique solution.

Problem 2 Consider the linear system $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$

a) Let $\mathbf{A}(t)$ be continuous for $-\infty < t < \infty$ and be periodic with period 2π. Show that if $\mathbf{\Phi}(t)$ is a fundamental matrix of the system for $-\infty < t < \infty$, then so is $\mathbf{\Phi}(t + 2\pi)$. Thus, prove that $\mathbf{\Phi}(t + 2\pi) = \mathbf{\Phi}(t)\mathbf{C}$ for some nonsingular matrix $\mathbf{C}$.

b) Let $\mathbf{A}(t)$ be continuous on some interval I. Show that the transformation $\mathbf{y} = \mathbf{T}(t)\mathbf{z}$, where $\mathbf{T}$ is a nonsingular differentiable matrix function on the interval I, reduces the system to the system

$$\mathbf{z}' = (\mathbf{T}^{-1}(t)\mathbf{A}(t)\mathbf{T}(t) - \mathbf{T}^{-1}(t)\mathbf{T}'(t))\,\mathbf{z}$$

Solution:

a) Since $\Phi(t)$ is a fundamental matrix of the System for $-\infty < t < \infty$, we have

$$|\Phi(t)| \neq 0.$$

So,

$$|\Phi(t+2\pi)| \neq 0 \quad \cdots\cdots \quad (1)$$

We also have

$$\Phi'(t) = A(t)\,\Phi(t), \quad -\infty < t < \infty$$

So,

$$\Phi'(t+2\pi) = A(t+2\pi)\,\Phi(t+2\pi)$$

$$\Rightarrow \Phi'(t+2\pi) = A(t)\,\Phi(t+2\pi) \quad \cdots\cdots (2)$$

Hence, $\Phi(t+2\pi)$ is a fundamental matrix of the system

Floquet's theorem states that there exists a nonsingular matrix $P(t)$ s.t $P(t+2\pi) = P(t)$, and a constant matrix R such that $\Phi(t) = P(t)\,e^{tR}$.

We have

$$\Phi(t+2\pi) = P(t+2\pi)\,e^{(t+2\pi)R}$$

$$= P(t)\,e^{tR}\,e^{2\pi R} = \Phi(t)\,e^{2\pi R}$$

Clearly $C = e^{2\pi R}$ is a non singular matrix.

b) $y = T(t) z$

$$\Rightarrow \quad y' = T'(t) z + T(t) z'$$

$$\Rightarrow \quad A(t) y = T'(t) z + T(t) z'$$

$$\Rightarrow \quad A(t) T(t) z = T'(t) z + T(t) z'$$

$$\Rightarrow \quad T(t) z' = A(t) T(t) z - T'(t) z$$

So, we have

$$z' = \left[T^{-1}(t) A(t) T(t) - T^{-1}(t) T'(t) \right] z$$

Problem 3 Consider the system

$$x' = 3x - 2x^2 - xy$$

$$y' = 3y - xy - 2y^2$$

a) Classify the critical points (0,0) and (1,1).
b) Show that the system is almost linear near the critical point (1,1).
c) Investigate the stability of the critical points.

Solution

a) The Jacobian matrix of the system is

$$J(x,y) = \begin{pmatrix} 3-4x-y & -x \\ -y & 3-x-4y \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \Rightarrow \text{ The origin is a proper node}$$

$$J(1,1) = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \Rightarrow \text{ Eigenvalues : } -3, -1$$

(1,1) is an improper node.

b) Put $u = x-1$, $v = y-1$. We obtain

$$\begin{cases} \dfrac{du}{dt} = \dfrac{dx}{dt} = x(3-2x-y) = (u+1)(3-2(u+1)-(v+1)) \\ \qquad\qquad = (u+1)(-2u-v) \\ \dfrac{dv}{dt} = \dfrac{dy}{dt} = y(3-x-2y) = (v+1)(3-(u+1)-2(v+1)) \\ \qquad\qquad = (v+1)(-u-2v) \end{cases}$$

$$\Rightarrow \quad \frac{du}{dt} = (-2u-v) - u(2u+v)$$

$$\frac{dv}{dt} = (-u-2v) - v(u+2v)$$

Let $F(u,v) = \begin{pmatrix} -u(2u+v) \\ -v(u+2v) \end{pmatrix}$

$$|F(u,v)| = |u|\,|2u+v| + |v|\,|u+2v|$$

$$\leq 2|u|(|u|+|v|) + 2|v|(|u|+|v|) \leq 2(|u|+|v|)^2$$

$$\Rightarrow \lim_{(|u|+|v|)\to 0} \frac{|F(u,v)|}{|u|+|v|} = 0 \Rightarrow \text{ the system is almost linear at } (1,1)$$

c) Eigenvalue of $J(0,0)$ is 3 (multiplicity 2).
Thus, $(0,0)$ is an unstable critical point.

Eigenvalues of $J(1,1)$ are both negative,
$\lambda_1 = -3$, $\lambda_2 = -1$.
Hence, $(1,1)$ is an asymptotically stable critical point.

Problem 4 Consider the system

$$x' = \left(-1 + \frac{1}{t+1}\right)x$$

$$y' = (-2 + e^{-t})y$$

Show that the zero solution is globally asymptotically stable.

Solution: Let $Y = \begin{pmatrix} x \\ y \end{pmatrix}$. We have

$$Y' = [A + B(t)]Y,$$

where

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad B(t) = \begin{pmatrix} \frac{1}{t+1} \\ e^{-t} \end{pmatrix}$$

$$\lim_{t \to \infty} B(t) = 0$$

Since all eigenvalues of A are negative

and $\lim_{t \to \infty} B(t) = 0$,

The zero solution of the system

is globally asymptotically stable.

Problem 5 Consider the system

$$x' = -x - x^2 - 3y$$

$$y' = 3x - y$$

Estimate the region of asymptotic stability of the zero solution of the system

Solution: critical points: $\left.\begin{aligned} -x - x^2 - 3y &= 0 \\ 3x - y &= 0 \end{aligned}\right] \Rightarrow y = 3x$

$\Rightarrow -x - x^2 - 9x = 0 \Rightarrow -x^2 - 10x = 0 \Rightarrow x = 0$ or $x = -10$

$O(0,0)$ [asymptotically stable]

$A(-10,-30)$ [unstable]

Consider the lyapunov function: $V = \frac{1}{2}(x^2+y^2)$

$\Rightarrow V^* = -x^2 - y^2 - x^3 = -[x^2(x+1) + y^2]$

$\Rightarrow -V^*$ is positive definite if $x > -1$.

Let $D = \{(x,y) \mid x > -1\}$. Both V and $-V^*$ are positive definite on D.

Let $E = \{(x,y) \in D \mid V^*(x,y) = 0\}$. E contains the origin. Since A is not in E, the origin is the only invariant set in E.

Note that the point $(-1,0)$ is on the boundary of D.

The curve $V(x,y) = V(-1,0) = \frac{1}{2}$ meets the boundary of D.

$$C_{1/2} = \{(x,y) \mid V(x,y) < \tfrac{1}{2}\}$$

$$= \{(x,y) \mid x^2+y^2 < 1\}$$

is included in the region of asymptotic stability of the zero solution.

Problem 6 Use the Lyapunov's second method to determine the stability of the zero solution of the system.

a) $\begin{cases} y_1' = -y_1^3 + y_1 y_2^2 \\ y_2' = -2y_1^2 y_2 - y_2^3 \end{cases}$

b) $\begin{cases} x' = y \\ y' = -y - 2x + 4x^3 \end{cases}$

Solution:

a) $$\begin{cases} y_1' = -y_1^3 + y_1 y_2^2 & \times\ y_1 \\ y_2' = -2y_1^2 y_2 - y_2^3 & \times\ y_2 \end{cases}$$

$$\begin{cases} \frac{1}{2}\frac{d}{dt} y_1^2 = -y_1^4 + y_1^2 y_2^2 \\ \frac{1}{2}\frac{d}{dt} y_2^2 = -2y_1^2 y_2^2 - y_2^4 \end{cases} \quad +$$

$$\frac{1}{2}\frac{d}{dt}(y_1^2 + y_2^2) = -y_1^4 - y_1^2 y_2^2 - y_2^4$$

$$V(y) = \frac{1}{2}(y_1^2 + y_2^2) \Rightarrow \quad \overset{*}{V}(y) = -y_1^4 - y_1^2 y_2^2 - y_2^4$$

V and $-V^*$ are both positive definite.

Hence, the zero solution is asymptotically stable.

Since $V(y) \to \infty$ as $\|y\| \to \infty$, the zero solution is globally asymptotically stable.

b) Let $g(x) = -4x^3 + 2x \Rightarrow \int_0^x g(s)\,ds = x^2 - x^4$, $\quad |x| \le \frac{1}{8}$

$$g^2(x) \le 4(x^2 - x^4)$$

x' times $y' = -y - g(x) \Rightarrow \frac{1}{2}\frac{d}{dt} y^2 = -y^2 - \frac{d}{dt}\left(\int_0^x g(s)\,ds\right)$

$$\frac{d}{dt}\left(\frac{y^2}{2} + x^2 - x^4\right) = -y^2$$

$$\beta \frac{d}{dt}\left(y\,g(x)\right) = \beta y' g(x) + \beta y x' g'(x)$$
$$= -\beta y g(x) - \beta g^2(x) + \beta y^2 g'(x)$$

$$V(x,y) = \frac{y^2}{2} + x^2 - x^4 + \beta y g(x) \Rightarrow \quad \overset{*}{V} = -y^2 - \beta g^2(x) - \beta y g(x) + \beta y^2 g'(x)$$

$$|g'(x)| \le \max_{|x| \le 1/8} |g'(x)| \equiv M$$

$$\Rightarrow \quad \beta y^2 g'(x) \le \beta M y^2$$

$$-V^* = y^2 + \beta g^2(x) + \beta y g(x) - \beta y^2 g'(x)$$

$$\ge y^2 + \beta g^2(x) - \beta\left(\gamma y^2 + \frac{1}{\gamma} g^2(x)\right) - \beta M y^2$$

$$\ge \left(1 - \gamma\beta - M\beta\right) y^2 + \beta\left(1 - \frac{1}{\gamma}\right) g^2(x)$$

$$V = \frac{y^2}{2} + x^2 - x^4 + \beta y g(x)$$

$$\ge \frac{y^2}{2} + x^2 - x^4 - \beta\left(\alpha y^2 + \frac{1}{\alpha} g^2(x)\right)$$

$$\ge \frac{y^2}{2} + x^2 - x^4 - \beta\left(\alpha y^2 + \frac{4}{\alpha}(x^2 - x^4)\right)$$

$$\ge \left(\frac{1}{2} - \alpha\beta\right) y^2 + \left(1 - \frac{4\beta}{\alpha}\right)(x^2 - x^4)$$

We can choose α, γ, β such that both V and $-V^*$ are positive definite on $D = \{(x,y) \mid |x| \le \frac{1}{8}, |y| < \infty\}$

$\Rightarrow$ $(0,0)$ is asymptotically stable

Problem 7 For the following system, show that the zero solution is globally asymptotically stable.

a) $\begin{cases} x' = -x + \frac{1}{2}y \\ y' = -y + \frac{1}{2}x \end{cases}$

b) $\begin{cases} y_1' = -y_1 - y_2 \\ y_2' = y_1 - y_2 - y_2^3 \end{cases}$

Solution:

a) $x' = -x + \frac{1}{2}y \quad \times x \quad \Rightarrow \frac{1}{2}\frac{d}{dt}x^2 = -x^2 + \frac{xy}{2}$

$y' = -y + \frac{1}{2}x \quad \times y \quad \Rightarrow \frac{1}{2}\frac{d}{dt}y^2 = -y^2 + \frac{xy}{2}$ +

$\frac{1}{2}\frac{d}{dt}(x^2+y^2) = -x^2 - y^2 + xy$

$V = \frac{1}{2}(x^2+y^2)$

$-V^* = x^2 + y^2 - xy > x^2 + y^2 - \frac{1}{2}(x^2+y^2) = \frac{1}{2}(x^2+y^2)$

Hence, V and $-V^*$ are both positive definite on $\mathbb{R}^2$

and $V(x,y) \to \infty$ if $\|(x,y)\| \to \infty$

This implies that $(0,0)$ is globally asymptotically stable

b) $\begin{cases} y_1' = -y_1 - y_2 & \times y_1 \\ y_2' = y_1 - y_2 - y_2^3 & \times y_2 \end{cases} \Rightarrow$ $\frac{1}{2}\frac{d}{dt}y_1^2 = -y_1^2 - y_1y_2$

$\frac{1}{2}\frac{d}{dt}y_2^2 = y_1y_2 - y_2^2 - y_2^4$ +

$\frac{1}{2}\frac{d}{dt}(y_1^2 + y_2^2) = -y_1^2 - y_2^2 - y_2^4$

$\left.\begin{array}{l} V = \frac{1}{2}(y_1^2+y_2^2) \\ -V^* = y_1^2 + y_2^2 + y_2^4 \end{array}\right\} \Rightarrow$ V and $-V^*$ are positive definite on $\mathbb{R}^2$

and $V(x,y) \to \infty$ if $\|(x,y)\| \to \infty$

Hence, the zero solution is globally asymptotically stable.