King Fahd University of Petroleum and Minerals
Department of Mathematics

MATH 435 - Exam 1 - Term 241

Duration: 120 minutes

Name:____

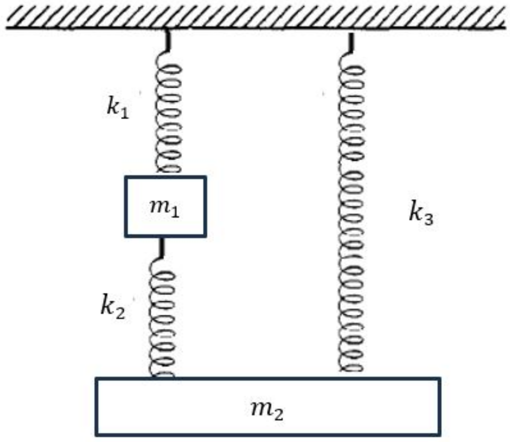
ID Number:____________________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 13 pages of problems. (Total of 6 Problems)

Question #	**Points**	**Maximum Points**
1		20
2		15
3		15
4		15
5		15
6		20
Total		100

1. **[10+5+5 points]** **(a)** Write a system of differential equations (equation of motion) for the following mass-spring system considering the air resistance.



(b) Write a system of first-order differential equations equivalent to the system in **(a)**.
(c) Write the vector-matrix notation of the first-order system of part **(b)**.

2. [**15 points**] Discuss the problem of existence and uniqueness of solutions of the initial value problem

$$\begin{cases} y_1' = e^t y_1 + \ln(t^2+1) y_3, \\ y_2' = t^3 y_1 - \cos(t) y_2 + \dfrac{1}{1-y_3^2}, \\ y_3' = y_1 + y_2^3, \end{cases}$$

$y_1(t_0) = \eta_1$, $y_2(t_0) = \eta_2$, $y_3(t_0) = \eta_3$.

3. [**4+7+4 points**] **(a)** Discuss the problem of existence and uniqueness of solutions of the initial value problem

$$y' = 3t(y-1)^{1/3}, \quad y(t_0) = y_0.$$

(b) Find the unique solution of the above IVP with the initial condition $y(0) = 2$.
(c) Find **two** solutions satisfying the differential equation and the initial condition $y(0) = 1$. Does this contradict what you have concluded in part **(a)**?

4. **[15 points]** Consider the initial value problem

$$y' + ay = g(t)y, \quad y(0) = y_0,$$

where a is a positive constant and g is a continuous function such that $\displaystyle\int_0^\infty |g(s)|ds < \infty$. Show that the solution is bounded, that is, there exists a non-negative constant C such that $|y(t)| \le C$ for all $t \ge 0$.

5. [**15 points**]
Solve the initial value problem below using the matrix exponential $e^{\boldsymbol{A}t}$.

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}, \quad \boldsymbol{y}(\boldsymbol{0}) = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix},$$

where $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix}$.

6. [**20 points**] Consider the system $\boldsymbol{y}' = \boldsymbol{A}(t)\boldsymbol{y}$ where

$$\boldsymbol{A}(t) = \begin{pmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{pmatrix}.$$

(a) Show that

$$\boldsymbol{\Phi}(t) = \begin{pmatrix} t^{-1} & t \\ -t^{-2} & 1 \end{pmatrix}$$

is a fundamental matrix for the system $\boldsymbol{y}' = \boldsymbol{A}(t)\boldsymbol{y}$ on any interval not containing the origin.

(b) Find the solution $\boldsymbol{\phi}(t)$ of the nonhomogeneous system

$$\boldsymbol{y}' = \boldsymbol{A}(t)\boldsymbol{y} + \begin{pmatrix} t \\ 1 \end{pmatrix}$$

satisfying the initial condition $\boldsymbol{\phi}(1) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.