King Fahd University of Petroleum and Minerals
Department of Mathematics

MATH 435 - Exam 2 - Term 241

Date: Sunday 24 November 2024.

Duration: 120 minutes

Name:____

ID Number:________________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write in legible.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 13 pages of problems. (Total of 5 Problems)

Question #	**Points**	**Maximum Points**
1		28
2		21
3		21
4		15
5		15
Total		100

1. [**4+8+6+3+7 points**] Consider the differential equation $x'' - x' - 2x = 0$.
(a) Write the equivalent system of first-order equations.
(b) Determine the canonical form J of the matrix in the system in part **(a)** and find the corresponding matrix T.
(c) Draw the phase portrait of the canonical system $z' = Jz$.
(d) Determine whether the origin is a node, saddle point, spiral point, or center. Is the origin attractor?
(e) Use the change of variable $y = Tz$ to find a fundamental matrix of the system in part **(a)**.

2. [**8+5+8 points**] Consider the initial value problem

$$\begin{cases} y_1' = y_1^2 + y_2 + 1 \\ y_2' = y_1 - y_2^2 + 1 \end{cases}, \quad y_1(0) = 0, \quad y_2(0) = 0$$

defined over the region $D = \left\{(t, y_1, y_2) \in \mathbb{R}^3 : |t-1| \leq 2, |y_1 - 1| \leq 3, |y_2 - 3| \leq 4\right\}$.
(a) Show that the IVP has a unique solution $\phi(t)$, $t \in I$. Give explicitly an interval I.
(b) Maximize the interval I in part **(a)**. Justify your answer.
(c) Apply Picard's iteration (make at least 3 successive approximations; $\phi_0(t)$, $\phi_1(t)$ and $\phi_2(t)$).

3. **[5+8+4+4 points]** Consider the 2π-periodic system $\boldsymbol{y}' = \boldsymbol{A}(\boldsymbol{t})\boldsymbol{y}$ where

$$\boldsymbol{A}(\boldsymbol{t}) = \begin{pmatrix} -1 & \cos t \\ -\cos t & -1 \end{pmatrix}$$

(a) Verify that the function $\phi(t) = e^{-t} \begin{bmatrix} \sin(\sin t) \\ \\ \cos(\sin t) \end{bmatrix}$ is a solution of the system.

(b) Find the characteristic multipliers and characteristic exponents of the system.

(c) Does the system have a periodic solution? Justify your answer.

(d) Does the nonhomogeneous system

$$\boldsymbol{y}' = \boldsymbol{A}(\boldsymbol{t})\boldsymbol{y} + \begin{pmatrix} 1 \\ \sin(t) \end{pmatrix}$$

have a periodic solution? Justify your answer.

4. **[15 points]** Let ϕ be a solution of the nth-order linear equation

$$u^{(n)} + a_1 u^{(n-1)} + \cdots + a_n u = f(t),$$

where a_1, $a_2, \cdots, a_n$ are constants and where f is continuous on $0 \le t < \infty$ and grows no faster than an exponential function, that is, there exit real constants $M > 0$, $T \ge 0$ and b such that

$$|f(t)| \le Me^{bt} \qquad \forall t \ge T.$$

Show that $\phi(t)$, $\phi'(t), \cdots, \phi^{(n)}(t)$ all grow no faster than an exponential function.

5. **[15 points]** Consider the system

$$\begin{aligned} x' &= g(x), \\ y' &= f(x)y, \\ x(t_0) = x_0, &\quad y(t_0) = y_0, \end{aligned}$$

where $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz and $f : \mathbb{R} \to \mathbb{R}$ is just continuous. Assume the existence of solutions and show only the uniqueness.