King Fahd University of Petroleum and Minerals
Department of Mathematics

MATH 435 - Final Exam - Term 241

Date: Tuesday 24 December 2024.
Duration: 150 minutes

Name:______________________________

ID Number:______________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write in legible.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 18 pages of problems. (Total of 8 Problems)

Question #	Points	Maximum Points
1		10
2		20
3		24
4		10
5		10
6		10
7		10
8		11
Total		105

Q1. [**5+5 points**] Discuss the existence and uniqueness of the solution to the following initial value problems.

(a)

$$y'' + \frac{\ln(t-1)}{t^2+1}y' + 3y = \sin(t), \quad y(t_0) = \beta_1,\ y'(t_0) = \beta_2.$$

(b)

$$\begin{aligned} x' &= 5xy - e^y + \tan^{-1}(x), \\ y' &= 2x - y + \sqrt{x^2y^2+1}, \\ x(t_0) &= \eta_1, \quad y(t_0) = \eta_2. \end{aligned}$$

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Q2. [**10+10 points**] Consider the system

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y},$$

where $\boldsymbol{A} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -2 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.

(a) Determine the stability of the zero solution.

(b) Find the unique solution that satisfies the system and the initial condition $\boldsymbol{y}(\boldsymbol{0}) = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$.

Q3. [**4+10+10 points**] Consider the system

$$x' = y$$
$$y' = -2y - x + 3xy + x^2(y+1)$$

(a) Find the two critical points of the system.
(b) Show that the system is almost linear (in the sense of Theorem 4.3) near each critical point found in part **(a)**.
(c) Determine the stability of each critical point.

Q4. **[10 points]** Use the Lyapunov second method to determine the stability of the zero solution of the system.

$$x_1' = -x_2 + x_1^3 + x_1 x_2^2$$
$$x_2' = x_1 + x_2^3 + x_2 x_1^2.$$

[**Hint**: Use the standard Lyapunov function]

Q5. **[10 points]** Use the Lyapunov second method to determine the stability of the zero solution of the system.

$$x' = xy - y^4$$
$$y' = -x^2 - y^3 + xy^3.$$

[**Hint**: Use the standard Lyapunov function]

Q6. **[10 points]** Use the Lyapunov second method to determine the stability of the zero solution of the system.

$$\begin{aligned} y_1' &= -2y_2 + y_2y_3 - y_1^3 \\ y_2' &= y_1 - y_1y_3 - y_2^3 \\ y_3' &= y_1y_2 - y_3^3 \end{aligned}$$

[**Hint**: Consider the Lyapunov function $V(\boldsymbol{y}) = y_1^2 + cy_2^2 + y_3^2$ where you need to determine a suitable positive value for c.]

Q7. [**10 points**] Show that the zero solution of the system below is globally asymptotically stable.

$$x' = -x + y^3 + y^5$$
$$y' = -x - y - xy^2.$$

[**Hint**: Use the Lyapunov second method where you multiply the first equation by x and the second equation by a suitable factor.]

Q8. **[11 points]** Consider the system

$$x' = y - xf(x,y)$$
$$y' = -x - yf(x,y),$$

where $f(x,y)$

1. satisfies the requirements for existence and uniqueness theorem,
2. $f(x,y) \not\equiv \pm 1$ (this condition is to assure that the only critical point is the origin),
3. $f(x,y) = 0$ only if $(x,y) = (0,0)$,
4. has a fixed sign.

(a) Discuss the stability of the zero solution.
(b) Give two examples of $f(x,y)$ (avoid the constant functions); one where the zero solution is asymptotically stable and another is unstable.