King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Math 437 - Major Exam I

AY 2022-2023 (Term 221)

Time Allowed: 120 Minutes

Name: .... **ID number:** ....

- Textbook, notes, mobiles and smart devices are not allowed in this exam.
- Write neatly and legibly. You may lose points for messy work.
- Show all your work. No points for answers without justification.

Question	Marks	Max Marks
1		15
2		20
3		25
4		20
5		10
6		10
Total		100

Question 1

Consider the linear second order PDE

$$2u_{xx} + 2u_{xt} - 4u_{tt} + x + t = 0$$

(a) Classify the PDE as being elliptic, parabolic or hyperbolic.

(b) Use the following change of variables

$$\xi = x - \frac{1}{2}t, \;\; \eta = x + t$$

and transform the PDE into its canonical form.

Question 2

(a) Find the eigenvalues and the eigenfunctions of the problem

$$X'' + \lambda X = 0; \;\; X(0) = X(L) = 0$$

(b) A thin, homogeneous bar of length π, thermal diffusivity 1, and insulated sides has its ends maintained at temperature zero. The bar has an initial temperature given by

$$f(x) = x \sin x$$

Determine the temperature distribution in the bar.

Question 3

Consider the nonhomogeneous initial-boundary value problem

$$\begin{cases} u_t = ku_{xx} + xt, \ \text{ for } \ 0 < x < \pi, t > 0 \\ u_x(0,t) = u_x(\pi,t) = 0, \ \text{ for } \ t > 0 \\ u(x,0) = 1, \ \text{ for } \ 0 < x < \pi \end{cases} \tag{1}$$

Derive a solution of (1), using that $u(x,t)$ will be in the form

$$\frac{1}{2}T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos(nx)$$

Question 4

If $u(x,t)$ is the solution of the initial-boundary value problem

$$\begin{cases} u_t = 2u_{xx} - 4u_x, \ \text{ for } \ 0 < x < 2, t > 0 \\ u(0,t) = 0, \ \ u(2,t) = 2e^{2-2t}, \ \text{ for } \ t > 0 \\ u(x,0) = f(x), \ \text{ for } \ 0 < x < 2 \end{cases}$$

and assume $u(x,t)$ is given by

$$u(x,t) = e^{\alpha x + \beta t}\left(v(x,t) + g(x)\right)$$

Choose α, β and $g(x)$ to obtain a standard heat equation ($v_t = 2v_{xx}$) for v. Define the initial-boundary value problem for v but do not solve it.

Question 5

If $u(x,t)$ is a continuous solution of the initial-boundary value problem

$$\begin{cases} u_t = 10u_{xx} + F(x,t), \;\; \text{for } \; 0 < x < 1, t > 0 \\ u(0,t) = e^t, u(1,t) = t, \;\; \text{for } \; t > 0 \\ u(x,0) = 1 - x, \;\; \text{for } \; 0 \le x \le 1 \end{cases}$$

Prove that $u(x,t)$ is a unique solution.

Question 6

(a) Show that $u(x,t) = Af(x-t) + Bg(x+t)$ is a solution of the wave equation $u_{tt} = u_{xx}$ for all x and t, where f and g are differentiable functions of a single variable, and $A, B \in \mathbb{R}$.

(b) Specify A, B and the functions f, g in (a) to find a solution of the initial-boundary value problem

$$\begin{cases} u_{tt} = u_{xx}, \ \text{ for } \ 0 < x < \pi, t > 0 \\ u(0,t) = -\sin t, \ \ u(\pi,t) = \sin t, \ \ t > 0 \\ u(x,0) = \sin x, \ \ u_t(x,0) = -\cos x, \ \ 0 < x < \pi \end{cases}$$