King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Math 437 - Major Exam II

AY 2022-2023 (Term 221)

Time Allowed: 120 Minutes

Name: .... **ID number:** ....

- Textbook, notes, mobiles and smart devices are not allowed in this exam.
- Write neatly and legibly. You may lose points for messy work.
- Show all your work. No points for answers without justification.

Question	**Marks**	**Max Marks**
1		15
2		15
3		20
4		15
5		20
6		15
Total		100

Question 1

Given the initial-boundary value problem for the wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & \text{for } 0 < x < L, t > 0 \\ u(0,t) = u(L,t) = 0, & \text{for } t > 0 \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), & \text{for } 0 < x < L \end{cases} \tag{1}$$

(a) Solve the IBVP (1) using separation of variables.

(b) Write the solution of the IBVP given the following information:

$$\phi(x) = 0, \psi(x) = x, c = 2, L = 1.$$

Question 2

Consider the IBVP:

$$\begin{cases} u_{tt} = 4u_{xx} + \cos x, & \text{for } 0 < x < \pi, t > 0 \\ u(0,t) = u(\pi,t) = 0, & \text{for } t > 0 \\ u(x,0) = u_t(x,0) = 0, & \text{for } 0 < x < \pi \end{cases}$$

Transform the equation into a homogeneous PDE for which you could apply the method of separation of variables.

Find that homogeneous PDE and define the initial and boundary conditions, but do not solve it.

Question 3

Consider Cauchy problem for the wave equation on the real line:

$$\begin{cases} u_{tt} = u_{xx}, & \text{for } -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), & \text{for } -\infty < x < \infty \end{cases}$$

(a) Derive d'Alembert's solution of the Cauchy problem (), starting with the fact that

$$u(x,t) = F(x-t) + G(x+t)$$

(b) Let $u_1(x,t)$ be the solution of the Cauchy problem with $\phi(x) = cos(x)$ and $\psi(x) = x$ and $u_2(x,t)$ be the solution of the Cauchy problem with $\phi(x) = cos(x) + \epsilon$ and $\psi(x) = x + \epsilon$.

Write the solutions $u_1(x,t)$ and $u_2(x,t)$, then show that $|u_1(x,t) - u_2(x,t)| \leq (1+T)\epsilon$ for all x and $0 \leq t \leq T$, for every positive number T.

Question 4

Solve the Dirichlet problem

$$\begin{cases} \nabla^2 u(x,y) = 0, & \text{for } 0 < x < 1, 0 < y < \pi \\ u(0,y) = u(1,y) = 0, & \text{for } 0 < y < \pi \\ u(x,0) = sin(\pi x),\ u(x,\pi) = 0, & \text{for } 0 < x < 1 \end{cases}$$

Question 5

Let D be a disk of radius ρ about the origin. Consider the Dirichlet problem on D using polar coordinates:

$$\begin{cases} \nabla^2 u(r,\theta) = 0, & \text{for } 0 \le r < \rho, -\pi \le \theta \le \pi \\ u(\rho,\theta) = f(\theta), & \text{for } -\pi \le \theta \le \pi \end{cases}$$

(a) Derive the Poisson's integral solution given that the solution of the Dirichlet problem is

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)d\xi + \sum_{n=1}^{\infty} \frac{r^n}{\pi\rho^n}\left[\int_{-\pi}^{\pi} f(\xi)\cos(n\xi)d\xi \cos(n\theta) + \int_{-\pi}^{\pi} f(\xi)\sin(n\xi)d\xi \sin(n\theta)\right]$$

(b) Solve the Dirichlet problem given the following information:

$$\rho = 7, u(7,\theta) = \cos^2\theta$$

Question 6

Solve the Neumann problem:

$$\begin{cases} \nabla^2 u(x,y) = 0, & \text{for } 0 < x < 1, 0 < y < 1 \\ u_x(0,y) = u_x(1,y) = 0, & \text{for } 0 < y < 1 \\ u_y(x,0) = 4\cos(\pi x), \ u_y(x,1) = 0, & \text{for } 0 < x < 1 \end{cases} \tag{2}$$

Define a necessary condition for the solution to exist. Does this problem have a unique solution?