King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Math 437 - Final Exam

AY 2022-2023 (Term 221)

Time Allowed: 180 Minutes

Name: .... **ID number:** ....

- Textbook, notes, mobiles and smart devices are not allowed in this exam.
- Write neatly and legibly. You may lose points for messy work.
- Show all your work. No points for answers without justification.

Question	**Marks**	**Max Marks**
1		20
2		25
3		20
4		25
5		20
6		20
7		10
Total		140

Question 1

Solve the Poisson problem

$$\begin{cases} u_{xx} + u_{yy} = \sqrt{3}y, & \text{for } 0 < x < \pi, 0 < y < 2\pi \\ u(x,0) = u(x,2\pi) = 0, & \text{for } 0 < x < \pi \\ u(0,y) = u(\pi,y) = 0, & \text{for } 0 < y < 2\pi. \end{cases}$$

Question 2

Consider the heat equation on the real line

$$\begin{cases} u_t = ku_{xx}, & \text{for } -\infty < x < \infty, t > 0 \\ u(x,0) = f(x), & \text{for } -\infty < x < \infty. \end{cases} \tag{1}$$

(a) Derive a solution for (1) using separation of variables and Fourier integral method.

(b) Reformulate the solution and verify that:

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} f(\xi)\, e^{-(x-\xi)^2/4kt} d\xi$$

(c) Write the solution of (1) given that

$$f(x) = \begin{cases} 1, & \text{for } -1 \le x \le 1, \\ 0, & \text{for } |x| > 1. \end{cases}$$

Then, expand the integrand of this solution in a Maclaurin series and integrate term by term to obtain a series solution for $u(x,t)$.

Question 3

Solve the Dirichlet problem for the upper half-plane

$$\begin{cases} \nabla^2 u = 0, & \text{for } -\infty < x < \infty, y > 0 \\ u(x,0) = f(x), & \text{for } -\infty < x < \infty. \end{cases}$$

given that

$$f(x) = \begin{cases} \cos(x), & \text{for } -\pi/2 \le x \le \pi/2, \\ 0, & \text{for } |x| > \pi/2. \end{cases}$$

Question 4

Consider the linear first-order partial differential equation:

$$3yu_x - 2xu_y = 0.$$

Determine the characteristics and the general solution of the partial differential equation. For each set of Cauchy data given, attempt to find a particular solution of the Cauchy problem. Determine if there is a unique solution, no solution, or infinitely many solutions.

(a) $u(x,y) = x^2$ on the line $y = x$.

(b) $u(x,y) = 1 - x^2$ on the line $y = -x$.

(c) $u(x,y) = 2x$ on the ellipse $2x^2 + 3y^2 = 4$.

Question 5

Find the solution of the quasilinear equation

$$yu_x - xu_y = e^u$$

that contains the curve Γ given by $y = \sin x, u = 0$

Question 6

Consider the initial boundary value problem (IBVP):

$$\begin{cases} u_t = 7u_{xx}, & \text{for } \ 0 < x < 5, t > 0 \\ u(0,t) = 1, \ u(5,t) = 4, & \text{for } \ t > 0 \\ u(x,0) = e^{-x}, & \text{for } \ 0 < x < 5. \end{cases} \tag{2}$$

Assume $u(x,t) = U(x,t) + \psi(x)$, where

$$U(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{5}\right) e^{-7n^2\pi^2 t/25}$$

satisfies the heat equation $U_t = 7U_{xx}$ for $0 < x < 5, t > 0$ and $U(0,t) = U(5,t) = 0$.

(a) Solve the IBVP (2) by finding c_n and $\psi(x)$.

(b) What does the solution $u(x,t)$ represent? Graph the solution for two different values of t. Specify the transient part and the steady-state part of the solution.

Question 7

Prove Duhamel's principle for the wave equation that is the solution of the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t), & \text{for } \ -\infty < x < \infty, t > 0 \\ u(x,0) = u_t(x,0) = 0, & \text{for } \ -\infty < x < \infty. \end{cases}$$

is

$$u(x,t) = \int_0^t w(x,t,\tau)d\tau$$

where $w = w(x,t,\tau)$ satisfies

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & \text{for } \ -\infty < x < \infty, t > \tau \\ w(x,\tau,\tau) = 0, \ w_t(x,\tau,\tau) = h(x,\tau), & \text{for } \ -\infty < x < \infty. \end{cases}$$