King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH441 - Advanced Calculus II
Exam I – Semester 222

Exercise 1

(a) Show that any open ball $B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| < r\}$ is open in $\mathbb{R}^n$.

(b) Let $(F_i)_{i\in I}$ be a collection of closed sets in $\mathbb{R}^n$. Show that $\cap_{i\in I} F_i$ is a closed subset of $\mathbb{R}^n$.

(c) Let S be a subset of $\mathbb{R}^n$. Show that $\partial S = \overline{S} \cap \overline{S^c}$. Deduce that ∂S is closed.

(d) Let S be an open set of $\mathbb{R}^n$, show that $(\partial S)^\circ = \emptyset$.
(Hint: the proof is by contradiction and use (c))

Exercise 2

Let (x_k) and (y_k) be two sequences in $\mathbb{R}^n$ with $x_k \to x$ and $y_k \to y$. Show that

(a) For $a, b \in \mathbb{R}$, show that $ax_k + by_k \to ax + by$.

(b) Show that $x_k \to x$ if and only if $x_k^i \to x^i$ for all $i = 1 \ldots n$.

Exercise 3

(a) Show that if F is a closed subset of S, a compact subset of $\mathbb{R}^n$, then F is compact.

(b) Show that a closed subset of a complete metric space is complete.

(c) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function and K is a compact set of $\mathbb{R}^n$. Using sequential compactness, show that $f(K)$ is a compact subset of $\mathbb{R}^m$.

Exercise 4

Let $T : U \to V$ be a linear transformation between two normed vector spaces.

(a) Define T is bounded.

(b) Show that if $U = \mathbb{R}^n$, then T is bounded.

(c) Show that T is bounded if and only if T is continuous.

Exercise 5

(a) Let $f : X \to Y$ be continuous between two metric spaces. Show if X is compact, then f is uniformly continuous on X.

(b) Show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

(c) Let $f : \Omega \to \mathbb{R}^m$ be a uniformly continuous on $\Omega \subset \mathbb{R}^n$. Show that

 (i) If (x_k) is a Cauchy sequence in Ω, then the sequence $(f(x_k))$ is Cauchy in $\mathbb{R}^m$.

 (ii) Show that if Ω is bounded then $f(\Omega)$ is bounded.
 (Hint: use that f can be extended continuously on $\overline{\Omega}$)

 (iii) Give an example of a continuous function f such that Ω is a bounded interval but $f(\Omega)$ is not bounded.

Exercise 6

Let

$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Show that f is continuous at $(0,0)$.

Exercise 7

(a) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous. Show that $f^{-1}(F)$ is closed in $\mathbb{R}^n$ for any F, a closed subset in $\mathbb{R}^m$.

(b) Define a homeomorphism between two metric spaces (X, d_X) and (Y, d_Y).

(c) Show that $f : [-\pi, \pi) \to S^1$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle, defined by $f(\theta) = e^{i\theta}$ is continuous, bijection but f^{-1} is not continuous on S^1.

(d) Let S be a compact subset of $\mathbb{R}^n$ and let $f : S \to \mathbb{R}^m$ be a continuous and one-to-one on S. Show that $f^{-1} : f(S) \to S$ is continuous.
(Hint use (a))