King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH441 - Advanced Calculus II
Exam 2 – Semester 222

Exercise 1

Let

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Show that f is continuous on $\mathbb{R}^2$.

(b) Find $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

(c) Show that f is not differentiable at $(0,0)$.

Exercise 2

Let

$$u = xyf(\frac{x+y}{xy}),$$

where $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function.

Show that u satisfies the partial differential equation

$$x^2 \frac{\partial u}{\partial x} - y^2 \frac{\partial u}{\partial y} = g(x, y)u,$$

and find g.

Exercise 3

Let

$$f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Examine the equality of $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

Exercise 4

Let $x \in \mathbb{R}^n$ and $x \neq 0$, let $f(x) = g(r)$, where $r = \|x\|$ and g is C^2 on $(0, \infty)$.

(a) Show that

$$\Delta f = g''(r) + \frac{n-1}{r} g'(r)$$

(b) Find all radial harmonic functions on $\mathbb{R}^n \setminus \{0\}$.

Exercise 5

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be $f(x, y) = (2ye^{2x}, xe^y)$ and $g : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $g(x, y) = (3x - y^2, 2x + y, xy + y^3)$.

(a) Show that there exists a neighborhood of $(0, 1)$ that f carries in a one-to-one fashion onto a neighborhood of $(2, 0)$.

(b) Find $D_{g \circ f^{-1}}(2, 0)$.

Exercise 6

(a) Show that the equations

$$xy^5 + yu^5 + zv^5 = 1$$

$$x^5y + y^5u + z^5v = 1$$

have a unique solution $(u, y) = f(x, y, z)$ near the point $(0, 1, 1, 1, 0)$

(b) Find the derivative $D_f(0, 1, 1)$.