King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH441 - Advanced Calculus II
Final Exam – Semester 222
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Exercise 1

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$f(x,y) = (e^x \cos y, e^x \sin y)$$

(a) Show that $J_f(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$.

(b) Show that f is *not* one-to one on $\mathbb{R}^2$.

(c) Let $S = \{(x,y) \in \mathbb{R}^2 : -\pi < y < \pi\}$. Show that the restriction of f on S is one-to-one and find its inverse.

(d) Find $f(S)$ and $f(\mathbb{R}^2)$.

Exercise 2

Show that the equations

$$xy^2 + xzu + yv^2 = 3,$$
$$u^3yz + 2xv - u^2v^2 = 2$$

have a unique solution $(u, v) = f(x, y, z) = (f_1(x, y, z), f_2(x, y, z))$ near the point $(1, 1, 1, 1, 1)$ and find $Df(1, 1, 1)$.

Exercise 3

Find the critical points of

$$f(x,y,z) = x^3 - y^3 + z^2 - 3x + 9y$$

and determine their nature.

Exercise 4

Let $f : R \to \mathbb{R}$ be integrable over a rectangle R. Show that

(i) If $f = 0$ a.e. then $\int_R f(x)dx = 0$.

(ii) If $f \geq 0$ and $\int_R f(x)dx = 0$, then $f = 0$ a.e.

Exercise 5

Evaluate the integral

$$\iint_\Omega \frac{2y}{\sqrt{1+(x+y)^3}} dxdy,$$

where

$$\Omega = \{(x,y) : x > 0, y > 0 \text{ and } x+y < 1\}$$

Hint: set $u = x + y$, $v = x - y$.

Exercise 6

Let Ω be the region in the first octant given by $x > 0, y > 0$ and $z > 0$ which is bounded by the plane $x + y + z = 1$. Use the change of variables

$$x = u(1 - v), y = uv(1 - w), z = uvw$$

to compute

$$\iiint_{\Omega} \frac{1}{y + z} dxdydz.$$

Exercise 7

Let c_n be the volume of the unit n-ball $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$

1. Show that
$$c_n = \frac{2\pi}{n} c_{n-2}, \text{ for } \quad n \geq 3.$$

2. Deduce the expression of c_n as a function of n.

3. Let $E^n = \left\{ x \in \mathbb{R}^n : \frac{x_1^2}{a_1^2} + \ldots + \frac{x_n^2}{a_n^2} \leq 1 \right\}$ be the n-dimensional solid ellipsoid. Show that the volume of E^n is given by $v(E^n) = a_1 \ldots . a_n c_n$.