King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH 441 - Semester 242
Midterm Exam

Exercise 1

1. Let (x_k) be a sequence in $\mathbb{R}^n$ such that $\|x_k - x_j\| < \dfrac{1}{k} + \dfrac{1}{j}$ for all integers j, k. Show that (x_k) converges.
2. Let $\{S_k : k = 1, 2, \ldots\}$ be a sequence of closed sets in $\mathbb{R}^n$ such that

$$S_{k+1} \subseteq S_k \quad \text{for every } k \quad \text{and} \quad d(S_k) \to 0 \text{ as } k \to \infty.$$

   Show that $\bigcap_{k=1}^{\infty} S_k = \{x\}$.

Exercise 2

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^\alpha\, y^\beta}{x^2 + xy + y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0), \end{cases}$$

   where $\alpha, \beta \in \mathbb{R}$. Under what condition on α and β is f continuous at $(0,0)$?

2. Consider the function $g(x,y) = \dfrac{1}{x^2 + y^2 - 1}$. Is g uniformly continuous in the open unit disk of $\mathbb{R}^2$?

Exercise 3

1. Let X be compact and $C(X)$ be the set of all continuous functions $f : X \to \mathbb{R}$. Show that $C(X)$ is a complete metric space under

$$d(f,g) \;=\; \|f-g\| \;=\; \sup_{x\in X}\big|f(x)-g(x)\big|.$$

2. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that T is bounded if and only if it maps bounded sets in $\mathbb{R}^n$ to bounded sets in $\mathbb{R}^m$.

Exercise 4

1. Let X be a metric space, show that any union of open sets is open.
2. Show that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if for each open set $V \subseteq \mathbb{R}^m$, the preimage $f^{-1}(V)$ is open in $\mathbb{R}^n$.
3. Suppose S is compact, and $f : S \to \mathbb{R}$ be continuous with $f(x) > 0$ for every $x \in S$. Show that there exists a constant $c > 0$ such that $f(x) > c$ for every $x \in S$.

Exercise 5

1. For $x \in \mathbb{R}^n \setminus \{0\}$, let $F(x) = f(r)$ where f is a C^2 function on $(0, \infty)$ and $r = \|x\|$. Show that

$$\frac{\partial^2 F}{\partial x_1^2} + \cdots + \frac{\partial^2 F}{\partial x_n^2} = f''(r) + (n-1)\frac{f'(r)}{r}.$$

2. Find all radial harmonic functions on $\mathbb{R}^n \setminus \{0\}$.

Exercise 6

1. Show that the function $f(x,y) = \sqrt{x\,y}$ is not differentiable at $(0,0)$.
2. Let $\alpha > \frac{1}{2}$. Show that the function $f(x,y) = |x\,y|^{\alpha}$ is differentiable at $(0,0)$.

Exercise 7

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = \big(x^2 + y,\ 2xy - y^2\big)$. Find the Jacobian $J_f(x,y)$.

2. Let $u = xy\, f\!\left(\frac{x+y}{xy}\right)$, where $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Show that u satisfies the partial differential equation

$$x^2 \frac{\partial u}{\partial x} - y^2 \frac{\partial u}{\partial y} = g(x,y),$$

and find $g(x,y)$.