King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH445 - Intro. to Complex Variables
Final Exam – Semester 221

Exercise 1

Let $f(z) = \dfrac{z+1}{z(z-2)^2}$.

(a) Find the Laurent series of f in $|z - 2| < 2$.

(b) Find the Laurent series of f in $0 < |z| < 2$.

Exercise 2

1. Classify the isolated singularities of $\dfrac{z^3+1}{z^2(z+1)}$

2. Compute
$$\oint_{|z|=4}\left[ze^{1/z^2}+\frac{\sin z}{z(z-\pi)^2}\right]dz.$$

Exercise 3

Prove that if f is analytic on and outside the simple closed contour Γ and has a zero of order 2 or more at ∞, then

$$\oint_\Gamma f(z)dz = 0.$$

Does this integral vanish if we merely assume that f has a simple zero at ∞.

Exercise 4

Define the function
$$h(z) = \frac{1}{\sin z} - \frac{1}{z} + \frac{2z}{z^2 - \pi^2}.$$

(a) Show that h is analytic on the disk $|z| < 2\pi$ except for the removable singularities at $z = 0, \pm\pi$.

(b) Find the first three terms of the Taylor series about $z = 0$ for $h(z)$. What is the radius of convergence of this series?

Exercise 5

1. Suppose that f has a pole of order m at the point z_0. Show that the function

$$g(z) = \frac{f'(z)}{f(z)}$$

   has a simple pole at z_0 with $Res(g, z_0) = -m$.

2. Let $n \geq 2$, use (1) to find

$$\oint_{|z|=2} \frac{z^{n-1}}{z^n - 1} dz.$$

Exercise 6

Let $a \in \mathbb{R}$ such that $|a| > 1$, find

$$\int_0^{2\pi} \frac{d\theta}{a - \sin\theta}$$

Exercise 7

Let $a, b \in \mathbb{R}$ such that $a \neq b$. Using the technique of residues, find

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{(x-a)(x-b)} \, dx$$