King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH445 - Introduction to Complex Variables
Major I Exam – Semester 241

Exercise 1

Prove that if $z \neq 1$, then

$$1 + z + z^2 + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}.$$

Use this result and De Moivre's formula to establish the following identities for $\theta \in (0, 2\pi)$:

(a) $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \dfrac{1}{2} + \dfrac{\sin\left(n + \frac{1}{2}\right)\theta}{2\sin\left(\frac{\theta}{2}\right)}$,

(b) $\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \dfrac{\sin(\frac{n\theta}{2})\sin\left((n+1)\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$,

Exercise 2

1. Find all values of $(1-\sqrt{3}i)^{\frac{1}{3}}$
2. Solve the equation $(z+1)^5 = z^5$.

Exercise 3

A real-valued function $u(x, y)$ satisfies

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2}$$

at every point in the annulus $A = \{z : 1 < |z| < 3\}$. Determine $u(x, y)$ up to an additive constant.

Exercise 4

Prove that $u(x, y) = x^3 - 3xy^2 + y$ is harmonic and find its harmonic conjugate.
Find an analytic function such that $\Re(f) = u$.

Exercise 5

Let $f(z) = e^z$.

(a) Describe the domain of definition and the range.

(b) Describe the image of the vertical line $\operatorname{Re} z = 1$.

(c) Describe the image of the horizontal line $\operatorname{Im} z = \frac{\pi}{4}$.

(d) Describe the image of the infinite strip $0 \leq \operatorname{Im} z \leq \frac{\pi}{4}$.

Exercise 6

Let $f(z) = u(x, y) + iv(x, y)$, where $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$. Prove that

$$\lim_{z \to z_0} f(z) = w_0$$

if, and only if,

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0.$$