King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH 445 - Introduction to Complex Variables
Major Exam II – Semester 241

Exercise 1

(a) Show that $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

(b) Solve $\cos z = \sin z$

(c) Show that the mapping $w = \sin z$ is one-to-one in the semi-infinite strip

$$S = \{z = x + iy : x \in (0, 2\pi) \text{ and } y > 0\}$$

and find the image of this strip.

Exercise 2

1. Find a branch of $\log(1-2z)$ that is analytic at all points in the plane except those on the following rays.
   (a) $\{x+iy|x \leq \frac{1}{2}, y=0\}$
   (b) $\{x+iy|x = \frac{1}{2}, y \geq 0\}$
2. Find a one-to-one analytic mapping of the upper half-plane Im $z > 0$ onto the infinite vertical strip $V := \{u+iv|0 < u < 1, -\infty < v < \infty\}$.

Exercise 3

1. Find all values of $(1+i)^{1+i}$.
2. Show that $\tanh^{-1} z = \dfrac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$.
3. Determine the inverse of the function $w = q(z) := 2e^z + e^{2z}$ explicitly in terms of the complex logarithm. Use your formula to find all values of z for which $q(z) = 3$.

Exercise 4

1. Show that if γ is the vertical line segment from $z = R (R > 0)$ to $z = R + 2\pi i$, then $|\int_{\gamma} \frac{e^{3z}}{1+e^z} dz| \leq \frac{2\pi e^{3R}}{e^R - 1}$
2. Show that the function $f(z) = \frac{1}{z}$ has no antiderivative in the punctured plane $\mathbb{C} \setminus \{0\}$.
3. Evaluate $\oint_{\Gamma} \frac{z}{(z+2)(z-1)} dz$ where Γ is the circle $|z| = r$ traversed twice in the clockwise direction.

Exercise 5

1. Evaluate
$$\int_\Gamma \frac{e^{iz}}{(z^2+1)^2}dz,$$
where Γ is the circle $|z| = 3$ traversed once counterclockwise.

2. Suppose that $f(z)$ is analytic at each point of the closed disk $|z| \leq 1$ and that $f(0) = 0$. Prove that the function
$$F(z) := \begin{cases} f(z)/z & z \neq 0, \\ f'(0) & z = 0, \end{cases}$$
is analytic on $|z| \leq 1$. **[HINT: To show that** F **is analytic at** $z = 0$ **note that the function**
$$G(z) := \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{f(\zeta)/\zeta}{\zeta - z}d\zeta$$
is analytic at this point. Using partial fractions deduce that $G(z) = F(z)$ **for** $|z| < 1$**.**

Exercise 6

1. Let f be entire and suppose that $\text{Re}\, f(z) \leq M$ for all $z \in \mathbb{C}$. Prove that f must be a constant function.
2. Suppose that f is entire and that $|f(z)| \leq |z|^3$ for all sufficiently large values of $|z|$, say $|z| > r_0$. Prove that f must be a polynomial of degree at most 3.