King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH 445 - Introduction to Complex Variables
Final Exam – Semester 241

Exercise 1

Find and state the convergence of the Taylor series of the following

(a) $\dfrac{1}{1+z}$ around $z_0 = 0$.

(b) $\dfrac{1+z}{1-z}$ around $z_0 = i$.

(c) $\dfrac{z}{(1-z)^2}$ around $z_0 = 0$.

Exercise 2

Define the function $h(z)$ by

$$h(z) = \frac{1}{\sin z} - \frac{1}{z} + \frac{2z}{z^2 - \pi^2}.$$

(a) Show that $h(z)$ is analytic in the disk $|z| < 2\pi$, except for removable singularities at $z = 0, \pm\pi$.

(b) Find the first three terms of the Taylor series about $z = 0$ for $h(z)$. What is the radius of convergence of this series?

Exercise 3

(a) Assume that $f(z)$ is analytic at the origin and that $f(0) = f'(0) = 0$. Prove that $f(z)$ can be written in the form $f(z) = z^2 g(z)$, where $g(z)$ is analytic at $z = 0$.

(b) Suppose that g is continuous on the circle $C : |z| = 1$, and that there exists a sequence of polynomials that converges uniformly to g on C. Prove that

$$\oint_C g(z)\, dz = 0.$$

(c) Find the first three nonzero terms in the Maclaurin expansion of $f(z) = \int_0^z e^{w^2}\, dw$.

Exercise 4

Find the Laurent series for the function $\dfrac{1}{(z+1)(z-2)}$ in each of the following domains:

(a) $|z| < 1$

(b) $1 < |z| < 2$

(c) $|z| > 2$

Exercise 5

(a) Prove that if $f(z)$ has a pole of order m at z_0, then $f'(z)$ has a pole of order $m+1$ at z_0.

(b) Prove that if $f(z)$ has a pole of order m at z_0, then $g(z) = \dfrac{f'(z)}{f(z)}$ has a simple pole at z_0. What is the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion for $g(z)$?

(c) What is the order of the zero at ∞ if $f(z)$ is a rational function of the form $\dfrac{P(z)}{Q(z)}$ with $\deg P < \deg Q$?

Exercise 6

(1) Evaluate each of the following integrals by means of the Cauchy residue theorem.

(a) $\oint_{|z|=5} \frac{\sin z}{z^2 - 4}\, dz$

(b) $\oint_{|z|=3} \frac{e^z}{z(z-2)^2}\, dz$

(2) Find p.v. $\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1}\, dx$

Exercise 7

(a) Find $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$

(b) Prove that all the roots of the equation $z^6 - 5z^2 + 10 = 0$ lie in the annulus $1 < |z| < 2$.