MATH 451 MIDTERM EXAM (TERM 211)

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1. Reparmetrize $\alpha(t) = (\cosh t,\, \sinh t,\, t)$ as a unit speed curve.

2. Prove that

$$\alpha(s) = \left(\frac{\cos^{-1} s - s\sqrt{1-s^2}}{2}, \frac{1-s^2}{2} \right), \quad -1 < s < 1$$

is a unit speed curve. Find $T(s)$, $N(s)$ and the curvature $\kappa(s)$. (Hint. Use $\frac{d}{ds}(\cos^{-1} s) = -\frac{1}{\sqrt{1-s^2}}$.)

3. Let α be a unit speed curve. Prove

$$(\alpha' \times \alpha'') \cdot \alpha''' = \kappa^2 \tau.$$

4. Show

$$\alpha(t) = (6t, 3t^2, t^3)$$

is a cylindrical helix, that is, there is a constant unit vector $\vec{u}$ whose angle with T is constant. What is the angle between $\vec{u}$ and T?

5. Let $\alpha(s)$ be a unit speed curve with constant curvature $\kappa \neq 0$. Suppose α is a cylindrical helix. Let $\vec{u}$ be the unit vector which has a constant angle θ with the tangent vector T of α. Assume by a change of coordinates that $\vec{u} = e_3 = (0,0,1)$ and let $\alpha(s) = (x(s), y(s), z(s))$. Then show that $\beta(s) = (x(s), y(s))$ is a circle. (s is not necessarily the arclength paramenter of β.) Express the radius of the circle in terms of κ and θ.

6. Let

$$\mathbf{x}(u,v) = (\cos u - v \sin u,\, \sin u + v\cos u,\, v)$$

be the parametrization of the hyperboloid $x^2 + y^2 - z^2 = 1$ as a ruled surface. Find the matrix representation of the shape operator S_p with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ at $p = \mathbf{x}(\frac{\pi}{2}, 0) = (0,1,0)$.

#1. $\alpha'(t) = (\sinh t, \cosh t, 1)$

$$\Rightarrow |\alpha'(t)| = \sqrt{\sinh^2 t + \cosh^2 t + 1}$$

$$= \sqrt{2\cosh^2 t} \qquad (1 + \sinh^2 t = \cosh^2 t)$$

$$= \sqrt{2}\cosh t.$$

$$\Rightarrow s(t) = \int_0^t |\alpha'(u)|\, du = \int_0^t \sqrt{2}\cosh u \, du$$

$$= \sqrt{2}\sinh t$$

$$\Rightarrow t = \sinh^{-1}\frac{s}{\sqrt{2}}$$

$$\Rightarrow \beta(s) = \alpha\left(\sinh^{-1}\frac{s}{\sqrt{2}}\right) = \left(\cosh\left(\sinh^{-1}\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}, \sinh^{-1}\frac{s}{\sqrt{2}}\right)$$

$$= \boxed{\left(\sqrt{1+\frac{s^2}{2}}, \frac{s}{\sqrt{2}}, \sinh^{-1}\frac{s}{\sqrt{2}}\right)}$$

$$\left(\cosh t = \sqrt{1+\sinh^2 t}. \Rightarrow \cosh\left(\sinh^{-1}\frac{s}{\sqrt{2}}\right) = \sqrt{1+\frac{s^2}{2}} \right)$$

#2 $\alpha(s) = \frac{1}{2}\left(\cos^{-1}s - s\sqrt{1-s^2},\ 1-s^2\right)$

$$\Rightarrow \alpha'(s) = \frac{1}{2}\left(-\frac{1}{\sqrt{1-s^2}} - \sqrt{1-s^2} + \frac{s^2}{\sqrt{1-s^2}},\ -2s\right)$$

$$= \frac{1}{2}\left(-\frac{1-s^2}{\sqrt{1-s^2}} - \sqrt{1-s^2},\ -2s\right)$$

$$= \left(-\sqrt{1-s^2},\ -s\right)$$

$\Rightarrow |\alpha'(s)|^2 = 1-s^2+s^2 = 1 \Rightarrow \alpha$ is of unit speed.

$$\Rightarrow T(s) = \alpha'(s) = \left(-\sqrt{1-s^2},\ -s\right)$$

$$T'(s) = \left(\frac{s}{\sqrt{1-s^2}},\ -1\right)$$

$$\Rightarrow |T'(s)|^2 = \frac{s^2}{1-s^2} + 1 = \frac{1}{1-s^2}$$

$$\Rightarrow k(s) = |T'(s)| = \frac{1}{\sqrt{1-s^2}} \quad \&$$

$$N(s) = \frac{1}{k(s)}T'(s) = \left(s,\ -\sqrt{1-s^2}\right)$$

#3

$$\alpha'(s) = T(s)$$

$$\alpha''(s) = T'(s) = -\kappa(s)N(s)$$

$$\alpha'''(s) = -\kappa'(s)N(s) - \kappa(s)N'(s)$$

$$= -\kappa'(s)N(s) - \kappa(s)\left(-\kappa(s)T(s) + \tau(s)B(s)\right)$$

$$= \kappa(s)^2 T(s) - \kappa'(s)N(s) - \kappa(s)\tau(s)B(s)$$

$$\Rightarrow \alpha' \times \alpha'' = -\kappa\, T \times N = -\kappa B$$

$$\Rightarrow (\alpha' \times \alpha'') \cdot \alpha''' = \kappa^2 \tau$$

#4 $\alpha(t) = (6t, 3t^2, t^3)$

$\Rightarrow \alpha'(t) = (6, 6t, 3t^2) = 3(2, 2t, t^2)$

$\alpha''(t) = 3(0, 2, 2t) = 6(0, 1, t)$

$\alpha'''(t) = 6(0, 0, 1)$

$\Rightarrow \alpha' \times \alpha'' = 18(t^2, -2t, 2)$

$\Rightarrow |\alpha'| = 3\sqrt{t^4+4t^2+4} = 3\sqrt{(t^2+2)^2} = 3(t^2+2)$

$|\alpha' \times \alpha''| = 18\sqrt{t^4+4t^2+4} = 18(t^2+2)$

& $(\alpha' \times \alpha'') \cdot \alpha''' = 18 \cdot 6 \cdot 2$

$\Rightarrow \kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{18(t^2+2)}{27(t^2+2)^3} = \frac{2}{3(t^2+2)^2}$

$\tau = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{|\alpha' \times \alpha''|^2} = \frac{18 \cdot 6 \cdot 2}{18^2 (t^2+2)^2} = \frac{2}{3(t^2+2)^2}$

$\Rightarrow \kappa = \tau \Rightarrow \underline{\underline{\frac{\tau}{\kappa} = 1}}$

therefore $\alpha(t)$ is a helix and if θ is the angle between $\vec{u}$ & T, then

$$\frac{\tau}{\kappa} = \cot\theta = 1.$$

$\Rightarrow \boxed{\theta = \pi/4}$

#5 $T = \alpha' = (x', y', z')$.

Since T & $\vec{u}$ make the constant angle θ,

$$T \cdot \vec{u} = \cos\theta = \text{const}$$

$$\shortparallel$$

$$(x', y', z') \cdot (0, 0, 1) = z'$$

$$\Rightarrow z(s) = s \cdot \cos\theta + c, \quad c \text{ a constant.}$$

Since α is of unit speed,

$$1 = (x')^2 + (y')^2 + (z')^2 = (x')^2 + (y')^2 + \cos^2\theta$$

$$\Rightarrow |\beta'(s)|^2 = (x'(s))^2 + (y'(s))^2 = 1 - \cos^2\theta = \sin^2\theta.$$

$$\Rightarrow |\beta'(s)| = \sin\theta \quad (\sin\theta \geq 0 \text{ since } 0 \leq \theta \leq \pi)$$

$$= \text{constant.}$$

i.e. β is a constant speed curve of speed $\sin\theta$.

$$\Rightarrow \gamma(s) = \beta\left(\frac{s}{\sin\theta}\right) = \left(x\left(\frac{s}{\sin\theta}\right), y\left(\frac{s}{\sin\theta}\right)\right) \text{ is}$$

of unit speed.

$$\Rightarrow \gamma''(s) = \frac{1}{\sin^2\theta}\left(x''\left(\frac{s}{\sin\theta}\right), y''\left(\frac{s}{\sin\theta}\right)\right)$$

$$\Rightarrow |\gamma''(s)| = \frac{1}{\sin^2\theta}\sqrt{\left(x''\left(\frac{s}{\sin\theta}\right)\right)^2 + \left(y''\left(\frac{s}{\sin\theta}\right)\right)^2}$$

$$= \frac{1}{\sin^2\theta}\left|\alpha''\left(\frac{s}{\sin\theta}\right)\right| = \frac{\kappa}{\sin^2\theta} = \text{curvature of } \beta$$

$\Rightarrow$ β is a planar curve of constant curvature

$$\frac{\kappa}{\sin^2\theta}$$

$\Rightarrow$ β is a circle & its radius is $\dfrac{\sin^2\theta}{\kappa}$.

#6 $\vec{X}(u,v) = (\cos u - v\sin u,\ \sin u + v\cos u,\ v)$

$$\Rightarrow \vec{X}_u = (-\sin u - v\cos u,\ \cos u - v\sin u,\ 0)$$
$$\vec{X}_v = (-\sin u,\ \cos u,\ 1)$$

$$\Rightarrow \vec{X}_u \times \vec{X}_v = (\cos u - v\sin u,\ \sin u + v\cos u,\ -v)$$

$$\Rightarrow |\vec{X}_u \times \vec{X}_v|^2 = (\cos u - v\sin u)^2 + (\sin u + v\cos u)^2 + v^2$$
$$= 1 + 2v^2$$

$$\Rightarrow \vec{n} = \frac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|}$$

$$= \frac{1}{\sqrt{1+2v^2}}(\cos u - v\sin u,\ \sin u + v\cos u,\ -v)$$

$$\Rightarrow \vec{n}_u = \frac{1}{\sqrt{1+2v^2}}(-\sin u - v\cos u,\ \cos u - v\sin u,\ 0)$$

$$= (-1,\ 0,\ 0) \quad \text{at } (u,v) = (\tfrac{\pi}{2}, 0)$$

$$\vec{n}_v = -\frac{1}{2}(1+2v^2)^{-3/2} \cdot 4v\ (\cos u - v\sin u,\ \sin u + v\cos u,\ -v)$$
$v=0$
$$+ \frac{1}{\sqrt{1+2v^2}}(-\sin u,\ \cos u,\ -1)$$

$$= (-1,\ 0,\ -1) \quad \text{at } (u,v) = (\tfrac{\pi}{2}, 0)$$

Since $\vec{X}_u = (-1, 0, 0)$, $\vec{X}_v = (-1, 0, 1)$

at $(u,v) = (\frac{\pi}{2}, 0)$,

$$\vec{n}_u = \vec{X}_u$$

$$\Rightarrow \quad \vec{n}_v = 2\vec{X}_u - \vec{X}_v ,$$

$$\Rightarrow \quad S\vec{X}_u = -dn(\vec{X}_u) = -\vec{n}_u = -\vec{X}_u$$

$$S\vec{X}_v = -dn(\vec{X}_v) = -2\vec{X}_u + \vec{X}_v$$

$\Rightarrow$ the matrix representation of S with respect to $\{\vec{X}_u, \vec{X}_v\}$ is

$$\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}.$$