EXAM I- MATH 453
Duration: 150 mn

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Exercise 1. Let X be a set and $\mathcal{CC}$ be the set of all $V \subseteq X$ such that either $V = \emptyset$ or $X \setminus V$ is countable.

(1) Show that $\mathcal{CC}$ is a topology on X (called the countable complement topology or the co-countable topology).
(2) If we let $\mathcal{CF}$ be the co-finite topology on X, then show that $\mathcal{CC}$ is finer than $\mathcal{CF}$.
(3) Show that if X is infinite, then $\mathcal{CC}$ is strictly finer than $\mathcal{CF}$.
(4) Show that the following statements are equivalent.
  (i) $\mathcal{CC} = \mathcal{CF}$.
  (ii) X is finite.
  (iii) $\mathcal{CF}$ is discrete.

Solution. **1**. By definition of $\mathcal{CC}$, $\emptyset \in \mathcal{CC}$. Also, as $X = X \setminus \emptyset$ and $\emptyset$ is countable, we deduce that $X \in \mathcal{CC}$.

- Let $U, V \in \mathcal{CC}$. If $U \cap V = \emptyset$, then $U \cap V \in \mathcal{CC}$. If we suppose that $U \cap V \neq \emptyset$, then $X = X \setminus U$ and $X = X \setminus V$ are countable. Hence $(X \setminus U) \cup (X \setminus V)$ is countable. So, by De Morgan's Law, $X \setminus (U \cap V)$ is countable. It follows that $U \cap V \in \mathcal{CC}$.

- Let $(U_i, i \in I)$ be an indexed family of elements of $\mathcal{CC}$. One may assume that $U_i \neq \emptyset$, for all $i \in I$ (as $\cup[U_i : i \in I] = \cup[U_i : i \in I$ and $U_i \neq \emptyset]$. As $X \setminus U_i$ is countable, for each $i \in I$, we deduce that $\bigcap_{i \in I}(X \setminus U_i)$ is countable. So, again, by De Morgan's Law, $X \setminus \left(\bigcup_{i \in I} U_i\right)$ is countable. We deduce that $\bigcup_{i \in I} U_i$ is in $\mathcal{CC}$.

As a result, $\mathcal{CC}$ is a topology on X.

2. Let $O \in \mathcal{CF}$. If $O = \emptyset$, then $O \in \mathcal{CC}$.

Now, if we assume $O \neq \emptyset$, then $X \setminus O$ is finite, so it is countable. It follows that $O \in \mathcal{CC}$. This shows that $\mathcal{CC}$ is finer than $\mathcal{CF}$.

3. Assume X is infinite; then there exists an infinite sequence $\{x_n : n \in \mathbb{N}\}$ ($x_n \neq x_m$, for m$\neq n$) of elements of X. The set $O = X \setminus \{x_{2n} : n \in \mathbb{N}\}$ is cocountable, but not co-finite. Hence $O \in \mathcal{CC}$ and $O \notin \mathcal{CF}$.

4. $(i) \Longrightarrow (ii)$. Suppose that X is infinite, and let $A \neq X$ be a countably infinite subset of X. Then A is $\mathcal{CC}$-closed, so it is $\mathcal{CF}$-closed. This implies that A is finite, a contradiction. As a result, X is finite.

$(ii) \Longrightarrow (iii)$. If X is finite, then for each $A \subseteq X$, $X - A$ is finite, and consequently A is $\mathcal{CF}$-open. Thus $\mathcal{CF}$ is discrete.

$(iii) \Longrightarrow (i)$. Assume $\mathcal{CF}$ is discrete. So, as

$$\mathfrak{D} = \mathcal{CF} \subseteq \mathcal{CC} \subseteq \mathfrak{D},$$

we deduce that $\mathfrak{D} = \mathcal{CF} = \mathcal{CC}$. □

Exercise 2. Let $(X, \mathcal{T})$ be a T_1-second countable topological space such that $|X| = 2^{\aleph_0}$. Show that $|\mathcal{T}| = 2^{\aleph_0}$.

Solution. Since X is a T_1-space $\{x\}$ is a closed set for every $x \in X$. The function

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{T} \\ x & \longmapsto & X \setminus \{x\} \end{array}$$

is obviously one-to-one, so $\left|X\right| = 2^{\aleph_0} \leq \left|\mathcal{T}\right|$.

Now, as X is second countable, it has a countable basis $\mathcal{B}$ (that is $\left|\mathcal{B}\right| \leq \aleph_0$).

Note that, if $(X, \mathcal{T})$ is a topological space and $\mathcal{B}$ is a basis of $\mathcal{T}$, then $\left|\mathcal{T}\right| \leq 2^{|\mathcal{B}|}$. Indeed, the function

$$\begin{array}{ccc} \mathcal{P}(\mathcal{B}) & \longrightarrow & \mathcal{T} \\ \Gamma & \longmapsto & \cup[O : O \in \Gamma] \end{array}$$

is onto. So$\left|\mathcal{T}\right| \leq \left|\mathcal{P}(\mathcal{B})\right| = 2^{|\mathcal{B}|}$.

We deduce that $\left|\mathcal{T}\right| \leq 2^{|\mathcal{B}|} \leq 2^{\aleph_0}$. Therefore $\left|\mathcal{T}\right| = 2^{\aleph_0}$. □

Exercise 3. Let X, Y be topological spaces and $A \subseteq X, B \subseteq Y$. Show that:

(1) $\overline{A \times B} = \overline{A} \times \overline{B}$.
(2) $\text{Int}\,(A \times B) = \text{Int}(A) \times \text{Int}(B)$.
(3) $(A \times B)^{'} = (A^{'} \times \overline{B}) \cup \left(\overline{A} \times B^{'}\right)$.
(4) $\text{Fr}\,(A \times B) = \left(\text{Fr}(A) \times \overline{B}\right) \cup \left(\overline{A} \times \text{Fr}(B)\right)$.

Solution. We denote by the complement of a subset A of a set X by A^c.

(1) Let $(x, y) \in X \times Y$, then the following equivalences hold:

$$\begin{aligned} (x, y) \notin \overline{A} \times \overline{B} &\iff x \notin \overline{A} \text{ or } x \notin \overline{B} \\ &\iff \left((x, y) \in (\overline{A})^c \times Y\right) \text{ or } \left((x, y) \in (X \times \overline{B})^c\right) \\ &\iff (x, y) \in ((\overline{A})^c \times Y) \cup (X \times (\overline{B})^c) \end{aligned}$$

It follows that $(\overline{A} \times \overline{B})^c = (X \times \overline{B}^c) \cup (\overline{A}^c \times Y)$ is open in the product topology. Consequently, $\overline{A} \times \overline{B}$ is a closed set containing $A \times B$. Thus, $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$.

Conversely, let $(x, y) \in \overline{A} \times \overline{B}$ and $U \times V$ be a basic open set of $X \times Y$, containing (x, y); then $U \cap A \neq \emptyset$ and $V \cap B \neq \emptyset$; hence $(U \times V) \cap (A \times B) \neq \emptyset$. This hows that $(x, y) \in \overline{A \times B}$.

As a result, $\overline{A \times B} = \overline{A} \times \overline{B}$.

(2) As $\text{Int}(A) \subseteq A$ and $\text{Int}(B) \subseteq B$, we deduce that $\text{Int}(A) \times \text{Int}(B) \subseteq A \times B$. Since $\text{Int}(A \times B)$ is the largest open set contained in $A \times B$, we deduce that $\text{Int}(A) \times \text{Int}(B) \subseteq \text{Int}(A \times B)$. Conversely, if $(x, y) \in \text{Int}(A \times B)$, then there exist an open set U of X and an open set V of Y, with $(x, y) \in U \times V \subseteq A \times B$. This So $x \in U \subseteq A$ and $y \in V \subseteq B$ Therefore $x \in \text{Int}(A)$ and $y \in \text{Int}(B)$.

As a result, we obtain the equality $\text{Int}\,(A \times B) = \text{Int}(A) \times \text{Int}(B)$.

(3) First, let us remark that if S is a subset of a topological space E, then $\overline{S} = A \cup S^{'}$, and consequently $S^{'} \subseteq \overline{S}$.

Let $(x, y) \in (A \times B)^{'} \subseteq \overline{A \times B} = \overline{A} \times \overline{B}$.

Suppose that $x \notin A^{'}$ and $y \notin B^{'}$; then there exist an open set U of X and an open set V of Y, such that $U \cap (A - \{x\}) = \emptyset$ and $V \cap (B - \{y\}) = \emptyset$. So $(U \times V) \cap [(A \times B) - \{(x, y)\}] = \emptyset$, this contradicts the fact that $(x, y) \in (A \times B)^{'}$.

We conclude that either $x \in A^{'}$ or $y \in B^{'}$, that is either $(x, y) \in A^{'} \times \overline{B}$ or $(x, y) \in \overline{A} \times B^{'}$. Therefore

$$(A \times B)^{'} \subseteq (A^{'} \times \overline{B}) \cup \left(\overline{A} \times B^{'}\right).$$

For the reverse containment, let $(x, y) \in A^{'} \times \overline{B}$, and $(x, y) \in O$ be an open set of $X \times Y$. So, there exist an open set U of X and an open set V of Y, with

$$(x, y) \in U \times V \subseteq O.$$

As $x \in A^{'}$, we deduce that $U \cap (A - \{x\}) \neq \emptyset$. Again, as $y \in \overline{B}$ and $y \in V$, we have $V \cap B \neq \emptyset$. Thus picking $z \in U \cap (A - \{x\})$ and $t \in V \cap B$, we obtain

$$(z, t) \in (U \times V) \cap [(A \times B) - \{(x, y)\}].$$

It follows that $O \cap ((A \times B) - \{(x, y)\}) \neq \emptyset$, showing that $(x, y) \in (A \times B)^{'}$.

An analogous argument shows that if $(x, y) \in \overline{A} \times B^{'}$, then $(x, y) \in (A \times B)^{'}$.

As a result, we obtain the equality:

$$(A \times B)^{'} = (A^{'} \times \overline{B}) \cup \left(\overline{A} \times B^{'}\right).$$

(4)

$$\begin{aligned}
\mathrm{Fr}(A \times B) &= \overline{A \times B} \cap \overline{(A \times B)^c} \\
&= (\overline{A} \times \overline{B}) \cap \overline{(X \times B^c) \cup (A^c \times Y)} \\
&= (\overline{A} \times \overline{B}) \cap \left(\overline{X \times B^c} \cup \overline{A^c \times Y} \right) \\
&= (\overline{A} \times \overline{B}) \cap \left((X \times \overline{B^c}) \cup (\overline{A^c} \times Y) \right) \\
&= \left[(\overline{A} \times \overline{B}) \cap (X \times \overline{B^c})\right] \cup \left[(\overline{A} \times \overline{B}) \cap (\overline{A^c} \times Y)\right] \\
&= \left[(\overline{A} \cap X) \times (\overline{B} \cap \overline{B^c})\right] \cup \left[(\overline{A} \cap \overline{A^c}) \times (\overline{B} \cap Y)\right] \\
&= (\overline{A} \times \mathrm{Fr}(B)) \cup (\mathrm{Fr}(A) \times \overline{B}).
\end{aligned}$$

□

Exercise 4. Let (X, d) be a metric space, $x \in X$ and r be a positive integer. Show that the following properties hold.

(1) The closed ball $\overline{B}_d(x, r) := \{y \in X : d(x, y) \leq r\}$ is a closed set of $(X, \mathcal{T}(d))$.
(2) $\overline{B_d(x, r)} \subseteq \overline{B}_d(x, r)$.
(3) The above containment can be strict.

Solution.

(1) We will show that $O = X \setminus \overline{B}_d(x, r) = \{y \in X : d(x, y) > r\}$ is $\mathcal{T}(d)$)-open. Indeed, let $y \in O$, then $d(x, y) > r$. We claim that $B_d(y, \varepsilon) \subseteq O$, for $\varepsilon \leq d(x, y) - r$.
To show this, Let $z \in B_d(y, \varepsilon)$. Then $d(z, y) < \varepsilon \leq d(x, y) - r$; that is $r < d(x, y) - d(z, y)$. But, according to the triangle inequality, we have $d(x, y) - d(z, y) \leq d(x, z)$. This leads to $r < d(x, z)$. It follows that $z \in O$, and consequently $B_d(y, \varepsilon) \subseteq O$. Therefore, O is $\mathcal{T}(d)$)-open.
(2) As $\overline{B}_d(x, r)$ is a $\mathcal{T}(d)$)-closed set containing $B_d(x, r)$, we conclude that $\overline{B_d(x, r)} \subseteq \overline{B}_d(x, r)$.
(3) Assume X is a set containing more than 1 element. We equip X with the discrete distance δ. Let $x \in X$. Then, $B_d(x, 1) = \{x\}$ and $\overline{B_d(x, 1)} = \overline{\{x\}} = \{x\}$, but $\overline{B}_d(x, 1) = X$. As a result $\overline{B_d(x, r)} \neq \overline{B}_d(x, r)$.

□

Exercise 5. The set $\mathbb{R}^n$ is equipped with the usual topology $\mathfrak{U}_n$. For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r) : \{y \in \mathbb{R}^n : \|x - y\| < r\}$, $\overline{B}(x, r) : \{y \in \mathbb{R}^n : \|x - y\| \leq r\}$ and $S(x, r) : \{y \in \mathbb{R}^n : \|x - y\| = r\}$, where $\|.\|$ is the Euclidean norm. Show that the following properties hold.

(1) $\overline{B(x, r)} = \overline{B}(x, r)$.
(2) $\text{Int}(\overline{B}(x, r)) = B(x, r)$.
(3) $\text{Fr}(B(x, r)) = S(x, r)$.

Solution. Straightforward.

□

Exercise 6. Find the interior and topological closure of each of the following subsets o the spaces $(\mathbb{R}^n, \mathfrak{U}_n)$.

(1) $\mathbb{Q}$
(2) $\mathbb{R} \setminus \mathbb{Q}$
(3) $\{(x, y) \in \mathbb{R}^2 : 0 < x < 1, y = 0\}$
(4) $\{\frac{1}{n} : n \in \mathbb{N}\}$
(5) The unit circle of $\mathbb{R}^2$: $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Solution. Straightforward.

(1),(2) $\overline{\mathbb{R} \setminus \mathbb{Q}} = \overline{\mathbb{Q}} = \mathbb{R}$, $\mathrm{Int}(\mathbb{Q}) = \mathrm{Int}(\mathbb{R} \setminus \mathbb{Q})\emptyset$.

(3) As $A = \{(x, y) \in \mathbb{R}^2 \colon 0 < x < 1, y = 0\} = (0, 1) \times \{0\}$, we have $\overline{A} = \overline{(0, 1)} \times \overline{\{0\}} = [0, 1] \times \{0\}$. However $\mathrm{Int}(A) = Int((0, 1)) \times \mathrm{Int}(\{0\}) = \mathrm{Int}((0, 1)) \times \emptyset = \emptyset$.

(4) If $A = \{\frac{1}{n} \colon n \in \mathbb{N}\}$, the $\overline{A} = A \cup \{0\}$ and $\mathrm{Int}(A) = \emptyset$.

(5) $\mathrm{Int}(S^1) = \emptyset$, $\overline{S^1} = S^1$.

□