**KFUPM-DEPARTMENT OF MATHEMATICS-MATH 453-EXAM II-TERM 231**

MATH 453: EXAM II, TERM (231), NOVEMBER 15, 2023

# EXAM II- MATH 453
**Duration: 150 mn**

**Student Name:**

**ID:**

**Exercise 1.** Let $X$ be an uncountable set. We equip $X$ with the co-countable topology $\mathcal{CC}$ (the open sets are: $\emptyset, X \setminus C$, with $C$ countable subsets of $X$), and $A$ be an infinite countable subset of $X$, show that the following properties hold.

(1) The convergent sequences in $(X, \mathcal{CC})$ are the stationary ones (recall that a sequence $(x_n)$ is said to be stationary if there exists $p \in \mathbb{N}$ such that $x_n = x_p$, for every $n \geq p$.
(2) $X - A$ is dense in $(X, \mathcal{CC})$.
(3) No element of $A$ is a limit of a sequence of elements of $X - A$.

***Solution.***

(1) Let $O$ be a nonempty $\mathcal{CC}$-open set of $X$; then $O = X \setminus C$, where $C$ is a countable subset of $X$. Assume $O \cap (X - A) = \emptyset$. Then $X = A \cup C$ is countable, a contradiction. We conclude that $X - A$ is $\mathcal{CC}$-dense in $X$.

(2) Let $(x_n)$ be a sequence of elements of $X$ converging to $\ell \in X$..
Suppose that there for every integer $p \in \mathbb{N}$, there exists $n > p$ such that $x_n \neq \ell$. Then there exists an integer $\varphi(1) > 1$ such that $x_{\varphi(1)} \neq \ell$. By induction on $n$, there exist integers $\varphi(n) > \varphi(n-1) > \ldots > \varphi(1) > 1$, with $x_{\varphi(n)} \neq \ell$, for every $n$. But as $X \setminus \{x_{\varphi(n)} \colon n \in \mathbb{N}\}$ is an open set containing $\ell$ and not containing $x_{\varphi(n)}$, for all $n$, we deduce that $(x_{\varphi(n)}, n \in \mathbb{N})$ does not converge to $\ell$. This contradicts the fact that a subsequence of a convergent sequence to $\ell$ converges to $\ell$. We conclude that there exists $p \in \mathbb{N}$, such that $x_n = \ell$, for all $n \geq p$. Conversely, it is clear that every stationary sequence is convergent.

(3) Let $x \in A$. As $\overline{X - A} = X$, $x$ is adherent to $X - A$. Any convergent sequence of elements of $X - A$ is stationary. So it converges to an element of $X - A$. Therefore, there is no sequence of elements of $X - A$ that converges to $x$.

□

**Exercise 2.** Let $X$ be a topological space, $Y$ be a Hausdorff space and $f, g \colon X \to Y$ be continuous maps. Show that the following properties hold.

(1) $\{x \in X | f(x) = g(x)\}$ is a closed set of $X$.
(2) If $D$ is dense in $X$ and $f_{|_D} = g_{|_D}$, then $f = g$.
(3) The graph $\text{Gr}(f) := \{(x, f(x)) : x \in X\}$ of $f \colon X \to Y$ is a closed set of $X \times Y$.
(4) If $f$ is injective and continuous, then $X$ is Hausdorff.
(5) Give an example of discontinuous function with closed graph.

***Proof.***

(1) Consider the function $\varphi\colon X \to Y \times Y$ defined by $\varphi(x) = (f(x), g(x))$; then as $p \circ \varphi = f$, and $q \circ \varphi = g$; (where $p$ is the first projection and $q$ the second projection), we deduce that $\varphi$ is continuous.
Now, as $\triangle_Y$ is closed and

$$C = \{x \in X\colon f(x) = g(x)\} = \varphi^{-1}(\triangle_Y), \text{ we deduce that } C \text{ is closed in } X.$$

(2) $C = \{x \in X\colon f(x) = g(x)\}$ is a closed set containing $D$; but as $D$ is dense we deduce that $\overline{C} \supseteq \overline{D} = X$, that $C = X$; therefore $f = g$.

(3) The graph of $f$ is given by $G(f)\colon = \{(x, y) \in X \times Y\colon y = f(x)\}$.
Let

$$\begin{array}{rccc} \psi\colon & X \times Y & \longrightarrow & Y \times Y \\ & (x, y) & \longmapsto & (f(x), y) \end{array}$$

Then, as $\psi$ is continuous and $G(f) = \psi^{-1}(\triangle_Y)$, we deduce that $G(f)$ is closed.

(4) Let $x \neq y$ be in $X$. As $f$ is injective $f(x) \neq f(y)$; so there exist two open sets $V_x, V_y$ of $Y$ such that.

$$f(x) \in V_x, \;\; f(y) \in V_y \;\text{ and } V_x \cap V_y = \phi.$$

Now, since $f$ is continuous, $f^{-1}(V_x)$, $f^{-1}(V_y)$ are disjoint open sets of $X$ containing $x$ and $y$, respectively. Therefore, $X$ is a $T_2$-space.

**An alternative proof.** Consider the function $\psi\colon X \times X \longrightarrow Y \times Y$ defined by $\psi(x, y) = (f(x), f(y))$. Let $U, V$ be open sets of $Y$. As $\psi^{-1}(U \times V) = f^{-1}(U) \times f^{-1}(V)$ and $f$ is continuous, we deduce that $\psi^{-1}(U \times V)$ is open in $X \times X$. Thus $\psi$ is a continuous function. Now, since $f$ is injective, it is clear that $\psi^{-1}(\Delta_Y) = \Delta_X$. Consequently, as $\Delta_Y$ is closed, we conclude that $\Delta_X$ is closed. As a result, $X$ is a $T_2$-space

□

**Exercise 3.** Let $X$ and $Y$ be topological spaces and $f\colon X \longrightarrow Y$ be function. Show that the following statements are equivalent.

(1) $f$ is continuous.
(2) For every $B \subseteq Y$, $f^{-1}(\operatorname{Int}(B)) \subseteq \operatorname{Int}(f^{-1}(B))$.
(3) For every $B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

***Solution***. $(i) \Longrightarrow (ii)$. The inverse image of an open set by a continuous function is open. So $f^{-1}(\operatorname{Int}(B))$ is an open set contained in $f^{-1}(B)$. Thus $f^{-1}(\operatorname{Int}(B)) \subseteq \operatorname{Int}(f^{-1}(B))$.

$(ii) \Longrightarrow (iii)$. Recall that, by duality, $\overline{B}^c = Int(B^c)$. Hence, applying $(ii)$ to the subset $B^c$, we obtain $f^{-1}(\operatorname{Int}(B^c)) \subseteq \operatorname{Int}(f^{-1}(B^c))$. By duality, we get $f^{-1}(\overline{B}^c) \subseteq \overline{f^{-1}(B))}^c$. But as $f^{-1}(\overline{B}^c) = \left(f^{-1}(\overline{B})\right)^c$, we deduce that $\left(f^{-1}(\overline{B})\right)^c \subseteq \overline{f^{-1}(B))}^c$. This leads to $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

$(iii) \Longrightarrow (i)$. Let $B$ be a closed set of $Y$, applying $(iii)$, we have $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) = f^{-1}(B)$. Consequently, $\overline{f^{-1}(B)} = f^{-1}(B)$, this means $f^{-1}(B)$ is a closed set of $X$. As a result $f$ is continuous. □

**Exercise 4.** Let $X$ be a topological space and $R$ be an equivalence relation on $X$. Show that the quotient space $X/R$ is a $T_1$-space if and only if $\overline{x}$ is closed in $X$, for each $x \in X$.

***Solution***. Let $p\colon X \longrightarrow X/R$ be the canonical onto map defined by

$$p(x) = \overline{x} = \{y \in X : xRy\}.$$

Assume $X/R$ is a $T_1$-space and $x \in X$. As $\{\overline{x}\}$ is closed in $X/R$ and $\overline{x} = p^{-1}(\{\overline{x}\})$, and $p$ is a continuous map, we deduce that $\overline{x}$ is closed in $X$. Conversely, suppose that $\overline{x}$ is closed in $X$. Then as $\overline{x} = p^{-1}(\{\overline{x}\})$, we deduce according to the quotient topology, that $\{\overline{x}\}$ is closed in $X/R$. This shows that $X/R$ is a $T_1$-space. □