MATH 453: EXAM II, TERM (232), MARCH 12, 2024

EXAM I- MATH 453
Duration: 120 mn

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Exercise 1. *Let X be a set and let $\mathbf{L}_1, \mathbf{L}_2 : \mathcal{P}(X) \to \mathcal{P}(X)$ be operators. Denote by $\mathbf{C} : \mathcal{P}(X) \to \mathcal{P}(X)$ the complement operator. Suppose that $\mathbf{L}_1 = \mathbf{C} \circ \mathbf{L}_2 \circ \mathbf{C}$; that is, $\mathbf{L}_1(A) = X \setminus \mathbf{L}_2(X \setminus A)$ for every $A \in \mathcal{P}(X)$. Prove that the following statements are equivalent:*

(1) *For all $A, B \subseteq X$, the following properties hold:*
  (a) $\mathbf{L}_1(\emptyset) = \emptyset$,
  (b) $A \subseteq \mathbf{L}_1(A)$,
  (c) $\mathbf{L}_1(\mathbf{L}_1(A)) = \mathbf{L}_1(A)$,
  (d) $\mathbf{L}_1(A \cup B) = \mathbf{L}_1(A) \cup \mathbf{L}_1(B)$.

(2) *For all $A, B \subseteq X$, the following properties hold:*
  (a) $\mathbf{L}_2(X) = X$,
  (b) $\mathbf{L}_2(A) \subseteq A$,
  (c) $\mathbf{L}_2(\mathbf{L}_2(A)) = \mathbf{L}_2(A)$,
  (d) $\mathbf{L}_2(A \cap B) = \mathbf{L}_2(A) \cap \mathbf{L}_2(B)$.

For a topological space $(X, \mathcal{T})$, with $\mathbf{Cl}$ and $\mathbf{Int}$ denoting the closure and interior operators respectively, show that $\mathbf{Cl} = \mathbf{C} \circ \mathbf{Int} \circ \mathbf{C}$.

Solution. As $\mathbf{C}$ is the complement operator, we have that

$$\mathbf{L}_1 = \mathbf{C} \circ \mathbf{L}_2 \circ \mathbf{C} \iff \mathbf{L}_2 = \mathbf{C} \circ \mathbf{L}_1 \circ \mathbf{C} \iff \mathbf{C} \circ \mathbf{L}_1 = \mathbf{L}_2 \circ \mathbf{C}.$$

$(1) \implies (2)$:

(1) $\mathbf{L}_2(X) = (\mathbf{C} \circ \mathbf{L}_1 \circ \mathbf{C})(X) = \mathbf{C} \circ \mathbf{L}_1(\emptyset) = \mathbf{C}(\emptyset) = X$.
(2) For any $A \subseteq X$, since $\mathbf{C}(A) \subseteq \mathbf{L}_1(\mathbf{C}(A)) = (\mathbf{L}_1 \circ \mathbf{C})(A) = (\mathbf{L}_2 \circ \mathbf{C})(A)$, it follows by applying $\mathbf{C}$ again (noting that $\mathbf{C}$ is decreasing) that $\mathbf{C}(\mathbf{L}_2(\mathbf{C}(A))) \subseteq A$, implying $\mathbf{L}_2(A) \subseteq A$.
(3) By operator composition, $\mathbf{L}_2 \circ \mathbf{L}_2 = (\mathbf{C} \circ \mathbf{L}_1 \circ \mathbf{C}) \circ (\mathbf{C} \circ \mathbf{L}_1 \circ \mathbf{C})$. Given that $\mathbf{C} \circ \mathbf{C} = \mathrm{Id}$, it simplifies to $\mathbf{L}_2 \circ \mathbf{L}_2 = \mathbf{C} \circ \mathbf{L}_1 \circ \mathbf{L}_1 \circ \mathbf{C} = \mathbf{C} \circ \mathbf{L}_1 \circ \mathbf{C} = \mathbf{L}_2$, using $\mathbf{L}_1 \circ \mathbf{L}_1 = \mathbf{L}_1$.
(4) For the property involving intersections, starting with the complement of $\mathbf{L}_2(A \cap B)$ gives $\mathbf{C}(\mathbf{L}_2(A \cap B)) = \mathbf{L}_1(\mathbf{C}(A \cap B)) = \mathbf{L}_1(\mathbf{C}(A) \cup \mathbf{C}(B)) = \mathbf{L}_1(\mathbf{C}(A)) \cup \mathbf{L}_1(\mathbf{C}(B)) = \mathbf{C}(\mathbf{L}_2(A)) \cup \mathbf{C}(\mathbf{L}_2(B)) = \mathbf{C}(\mathbf{L}_2(A) \cap \mathbf{L}_2(B))$. Applying $\mathbf{C}$ to both sides yields $\mathbf{L}_2(A \cap B) = \mathbf{L}_2(A) \cap \mathbf{L}_2(B)$.

The implication $(2) \implies (1)$ can be shown by analogous arguments.

For a topological space $(X, \mathcal{T})$, with $\mathbf{Cl}$ and $\mathbf{Int}$ denoting the closure and interior operators respectively, consider any subset $A \subseteq X$. Since $\mathbf{Int}(X - A) \subseteq X - A$, it follows that $A \subseteq X - (\mathbf{Int}(X - A))$. Taking closures, we have $\overline{A} \subseteq X - (\mathbf{Int}(X - A))$. Conversely, since $A \subseteq \overline{A}$, we deduce that $X - \overline{A} \subseteq X - A$ and $X - \overline{A} \subseteq \mathbf{Int}(X - A)$. Therefore, $X - \mathbf{Int}(X - A) \subseteq \overline{A}$. These inclusions prove that $\overline{A} = X - (\mathbf{Int}(X - A))$, which shows that $\mathbf{Cl} = \mathbf{C} \circ \mathbf{Int} \circ \mathbf{C}$. □

Exercise 2. Let d be the usual metric on $\mathbb{R}$ and $d' = \min(1, d)$. Show that d and d' are not Lipschitz equivalent.

Solution. Assume, for the sake of contradiction, that d and d' are Lipschitz equivalent. This means there exist two positive real constants α and β such that for all $x, y \in \mathbb{R}$, we have

$$\alpha d'(x, y) \leq d(x, y) \leq \beta d'(x, y).$$

Specifically, since $d' = \min(1, d)$, this inequality implies that $d(x, y) \leq \beta$, suggesting that d is bounded by β.

However, by the definition of d, for any $\beta > 0$, we can find $x, y \in \mathbb{R}$ such that $d(x, y) > \beta$. For instance, if we choose $x = \beta$ and $y = 3\beta$, then $d(x, y) = |x - y| = 2\beta > \beta$, which contradicts the assumption that $d(x, y) \leq \beta d'(x, y)$ for all x, y.

Therefore, d and d' cannot be Lipschitz equivalent. □

Exercise 3. Let (X, d) be a metric space, show that for all $x, y, z \in X$, we have:

$$|d(x, z) - d(x, y)| \leq d(y, z).$$

Solution.

- By the triangle inequality, we have:

  $$d(x, z) \leq d(x, y) + d(y, z),$$

  so $d(x, z) - d(x, y) \leq d(y, z)$.
- Again, using the triangle inequality, we get:

  $$d(x, y) \leq d(x, z) + d(z, y),$$

  which rearranges to $d(x, y) - d(x, z) \leq d(y, z)$.

  We conclude that

$$-d(y, z) \leq d(x, z) - d(x, y) \leq d(y, z).$$

Therefore,

$$|d(x, z) - d(x, y)| \leq d(y, z).$$

□

Exercise 4. Let d_1, d_2 be two distances on X.

(1) Show that $d = d_1 + d_2$ and $d' = \max(d_1, d_2)$ are distances on X.
(2) Show that d and d' are Lipschitz equivalent.

Solution.

(1) To verify that d and d' are distances on X, we must check that they satisfy the three distance axioms: non-negativity, identity of separation, and the triangle inequality.
   - For d, since both d_1 and d_2 are non-negative for all $x, y \in X$, their sum $d_1(x, y) + d_2(x, y)$ is also non-negative. If $x = y$, then $d_1(x, y) = d_2(x, y) = 0$, hence $d(x, y) = 0$. Conversely, if $d(x, y) = 0$, since d_1 and d_2 are both distances, it follows that $x = y$. The triangle inequality follows from the triangle inequalities for d_1 and d_2.

- For d', non-negativity and identity of separation follow similarly. The triangle inequality for d' uses the fact that $\max$ of two values is less than or equal to the sum of the same two values.

(2) To show that d and d' are Lipschitz equivalent, observe that for any $x, y \in X$,
$$d' \leq d \leq 2d'.$$
This is because $d' = \max(d_1, d_2) \leq d_1 + d_2 = d$ and $d_1 + d_2 \leq 2\max(d_1, d_2) = 2d'$. Therefore, for any x, y, it follows that $\frac{1}{2}d(x,y) \leq d'(x,y) \leq d(x,y)$, establishing the Lipschitz equivalence with Lipschitz constants $\alpha = 1$ and $\beta = 2$.

□

Exercise 5. Let us define the set
$$\mathfrak{LR} := \{V \subseteq \mathbb{R} : \forall x \in V, \exists a \in \mathbb{R} \text{ such that } x \in (-\infty, a) \subseteq V\}.$$

(1) Prove that $\mathfrak{LR}$ forms a topology on $\mathbb{R}$, referred to as the left ray topology.
(2) Demonstrate that
$$\mathfrak{LR} = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, a) : a \in \mathbb{R}\}.$$
(3) Establish that $\mathfrak{LR}$ is strictly coarser than the standard topology $\mathfrak{U}$ on $\mathbb{R}$.

Solution.

1. To show $\mathfrak{LR}$ forms a topology, we verify its properties:
- The empty set $\emptyset$ and $\mathbb{R}$ are in $\mathfrak{LR}$, as the conditions are vacuously true for $\emptyset$ and trivially satisfied for $\mathbb{R}$.
- The intersection of two elements in $\mathfrak{LR}$ remains in $\mathfrak{LR}$. Given $U, V \in \mathfrak{LR}$ and any $x \in U \cap V$, we find $a, b \in \mathbb{R}$ such that $x \in (-\infty, a) \subseteq U$ and $x \in (-\infty, b) \subseteq V$. Letting $c = \min(a, b)$, it follows $x \in (-\infty, c) \subseteq U \cap V$.
- Arbitrary unions of elements in $\mathfrak{LR}$ also belong to $\mathfrak{LR}$. For any $(U_i)_{i \in I}$ in $\mathfrak{LR}$ and $x \in \bigcup_{i \in I} U_i$, there exists a $j \in I$ and an $a \in \mathbb{R}$ such that $x \in (-\infty, a) \subseteq U_j$, implying $x \in (-\infty, a) \subseteq \bigcup_{i \in I} U_i$.

2. We show $\mathfrak{LR}$ is exactly $\{\emptyset, \mathbb{R}\} \cup \{(-\infty, a) : a \in \mathbb{R}\}$:
- Clearly, $\emptyset$ and $\mathbb{R}$ are in $\mathfrak{LR}$, and for any $a \in \mathbb{R}$, $(-\infty, a)$ meets the definition of $\mathfrak{LR}$.
- Conversely, any $V \in \mathfrak{LR}$ that is not $\emptyset$ or $\mathbb{R}$ must be of the form $(-\infty, a)$ for some $a \in \mathbb{R}$, by the definition of $\mathfrak{LR}$.

3. $\mathfrak{LR}$ is strictly coarser than the standard topology $\mathfrak{U}$ on $\mathbb{R}$:
- Every element of $\mathfrak{LR}$ is open in $\mathfrak{U}$, but there exist sets open in $\mathfrak{U}$, like $(0, 1)$, that cannot be a union of sets from $\mathfrak{LR}$.

□