EXAM II- MATH 453

Duration: 120 mn

Student Name:

ID:

Exercise 1. Consider the real line $\mathbb{R}$ equipped with the usual topology $\mathfrak{U}$:

(1) Give an example of real-valued sequences (x_n) such that $\operatorname{Lim}_{n\to+\infty}(x_n) \neq \operatorname{Adh}(x_n)$.

(2) Give an example of real-valued sequences (x_n) such that $\operatorname{Adh}(x_n) \neq \overline{\{x_n : n \in \mathbb{N}\}}$.

Solution.

(1) The sequence $x_n = (-1)^n$ in $(\mathbb{R}, \mathfrak{U})$ illustrates this scenario. For this sequence, $\operatorname{Lim}_{n\to+\infty}(x_n) = \emptyset$, $\operatorname{Adh}(x_n) = \{-1, 1\}$, showing that $\operatorname{Lim}_{n\to+\infty}(x_n) \neq \operatorname{Adh}(x_n)$.

(2) For the sequence $x_n = n$ in $(\mathbb{R}, \mathfrak{U})$, we find that $\operatorname{Adh}(x_n) = \bigcap_{n\in\mathbb{N}} \{k \colon k \geq n\} = \emptyset$. However, the closure of the sequence, $\overline{\{n : n \in \mathbb{N}\}} = \overline{\mathbb{N}} = \mathbb{N}$. Therefore, $\operatorname{Adh}(x_n) \neq \overline{\{x_n : n \in \mathbb{N}\}}$.

□

Exercise 2. Let X be a topological space and Y a Hausdorff space. Suppose $f, g : X \longrightarrow Y$ are continuous maps. Show that the following properties hold.

(1) The set $\{x \in X \mid f(x) = g(x)\}$ is closed in X.
(2) If D is dense in X and $f|_D = g|_D$, then $f = g$.
(3) The graph $\operatorname{Gr}(f) := \{(x, f(x)) \mid x \in X\}$ of f is closed in $X \times Y$.
(4) If f is injective and continuous, then X is Hausdorff.

Proof.

(1) Consider the function $\varphi : X \longrightarrow Y \times Y$ defined by $\varphi(x) = (f(x), g(x))$; then, as $p \circ \varphi = f$ and $q \circ \varphi = g$ (where p and q are the first and second projections, respectively), we deduce that φ is continuous (see Exercise **??**). Now, as Y is Hausdorff, the diagonal $\Delta_Y = \{(y, y) \mid y \in Y\}$ is closed in $Y \times Y$. Hence,

$$C = \{x \in X \mid f(x) = g(x)\} = \varphi^{-1}(\Delta_Y)$$

is closed in X.

(2) Since $C = \{x \in X \mid f(x) = g(x)\}$ is a closed set containing D and D is dense, we have $\overline{C} \supseteq \overline{D} = X$, implying $C = X$; therefore, $f = g$.

(3) The graph of f is given by $\operatorname{Gr}(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$.
Let

$$\begin{array}{rcl} \psi : X \times Y & \longrightarrow & Y \times Y \\ (x, y) & \longmapsto & (f(x), y). \end{array}$$

One may easily see that ψ is continuous. To see this, let U, V be open sets of Y. Then $\psi^{-1}(U \times V) = f^{-1}(U) \times V$, which is an open set of $X \times Y$. Now, as $\operatorname{Gr}(f) = \psi^{-1}(\Delta_Y)$, we deduce that $\operatorname{Gr}(f)$ is closed.

(4) Let $x \neq y$ in X. Since f is injective, $f(x) \neq f(y)$; thus, there exist two open sets V_x, V_y in Y such that

$$f(x) \in V_x,\ f(y) \in V_y, \text{ and } V_x \cap V_y = \emptyset.$$

Now, since f is continuous, $f^{-1}(V_x)$ and $f^{-1}(V_y)$ are disjoint open sets in X containing x and y, respectively. Therefore, X is a T_2-space (Hausdorff). □

Exercise 3. Let X be a topological space and $f : X \longrightarrow (\mathbb{R}, \mathfrak{U})$. Show that f is continuous if and only if, for each real number a, the sets $\{x \in X \mid f(x) > a\}$ and $\{x \in X \mid f(x) < a\}$ are open in X.

Solution. The collection $\Sigma = \{(-\infty, b) \mid b \in \mathbb{R}\} \cup \{(a, +\infty) \mid a \in \mathbb{R}\}$ forms a subbasis for the usual topology $\mathfrak{U}$ on $\mathbb{R}$. Hence, f is continuous if and only if $f^{-1}(U)$ is open in X for every $U \in \Sigma$.
Clearly, $\{x \in X \mid f(x) > a\} = f^{-1}((a, +\infty))$ and $\{x \in X \mid f(x) < b\} = f^{-1}((-\infty, b))$, for all $a, b \in \mathbb{R}$. Thus, the statement is proven. □

Exercise 4. Let A be an open subspace of $(\mathbb{R}, \mathfrak{U})$ which is not closed, and $\chi_A : \mathbb{R} \longrightarrow \mathbb{R}$ be the characteristic function of A defined by:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Show that χ_A is not continuous, however, all sets of the type $\{x \in \mathbb{R} \mid \chi_A(x) > b\}$ are open.

Solution. Since A is not closed, $\chi_A^{-1}((-\infty, 1)) = \{x \in \mathbb{R} \mid \chi_A(x) < 1\} = \mathbb{R} \setminus A$ is not open. So χ_A is not continuous.
For $b \in \mathbb{R}$, consider three cases:
Case 1: If $b \geq 1$, then $\chi_A^{-1}((b, +\infty)) = \{x \in \mathbb{R} \mid \chi_A(x) > b \geq 1\} = \emptyset$, which is open in $\mathbb{R}$.
Case 2: If $0 \leq b < 1$, then $\chi_A^{-1}((b, +\infty)) = \{x \in \mathbb{R} \mid \chi_A(x) = 1\} = A$, which is open in $\mathbb{R}$.
Case 3: If $b < 0$, then $\chi_A^{-1}((b, +\infty)) = \mathbb{R}$, which is open in $\mathbb{R}$. □

Exercise 5. Let X, Y be topological spaces and $f : X \longrightarrow Y$ be a function. Show that the following statements are equivalent.

(i) f is open.
(ii) $f(\mathbf{Int}(A)) \subseteq \mathbf{Int}(f(A))$, for each $A \subseteq X$.
(iii) f maps each member of a basis of X to an open set of Y.
(iv) For each $x \in X$ and each U in $\mathcal{N}(x)$, there exists V in $\mathcal{N}(f(x))$, with $V \subseteq f(U)$.

Solution.
$(i) \implies (ii)$. Suppose that f is an open map. As $\mathbf{Int}(A) \subseteq A$, we have $f(\mathbf{Int}(A)) \subseteq f(A)$. Since f is open, $f(\mathbf{Int}(A))$ is open in Y. Given that $\mathbf{Int}(f(A))$ is the largest open set contained in $f(A)$, it follows that $f(\mathbf{Int}(A)) \subseteq \mathbf{Int}(f(A))$.
$(ii) \implies (iii)$. Let $\mathcal{B}$ be a basis for the topology of X and let $U \in \mathcal{B}$. Then

$$f(\text{Int}(U)) = f(U) \subseteq \mathbf{Int}(f(U)) \subseteq f(U).$$

This implies $f(U) = \mathbf{Int}(f(U))$, so $f(U)$ is open in Y.
$(iii) \implies (iv)$. Given a basis $\mathcal{B}$ of X and an element $x \in X$, for any $U \in \mathcal{N}(x)$, there exists $V \in \mathcal{B}$ such that $x \in V \subseteq U$. Since $f(V)$ is open and contains $f(x)$, and $f(x) \in f(V) \subseteq f(U)$, it follows that $f(U) \in \mathcal{N}(f(x))$.
$(iv) \implies (i)$. For any open set U in X and any $y = f(x) \in f(U)$ with $x \in U$, since $U \in \mathcal{N}(x)$, there exists $V \in \mathcal{N}(f(x))$ with $V \subseteq f(U)$. Thus, $f(U)$ acts as a neighborhood of all its points, showing $f(U)$ is open. □

Exercise 6. Let $(X_i, i \in I)$ be a family of topological spaces. Denote by X_{box} and X_{prod} the set $X = \prod_{i\in I} X_i$ equipped with the box topology and the product topology, respectively.

(1) Show that the box topology on X is finer than the product topology.
(2) Let $\mathbb{R}^\omega_{\text{box}} = \prod_{n\in\mathbb{N}} X_n$ be equipped with the box topology, where X_n is the real line $\mathbb{R}$ equipped with the usual topology. Consider the function $f\colon (\mathbb{R}, \mathfrak{U}) \longrightarrow \mathbb{R}^\omega_{\text{box}}$ defined by $f(x) = (x, x, \ldots, x, \ldots)$. Show that f is not continuous, but $p_n \circ f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is the identity map, and hence continuous, where p_n is the n^{th}-projection.
(3) Let $(x_n, n \in \mathbb{N})$ be a sequence of elements of the product space $\prod_{i\in I} X_i$. Prove that (x_n) converges to $x \in X$ if and only if for each $i \in I$, the sequence $(p_i(x_n), n \in \mathbb{N})$ converges to $p_i(x)$, where $p_i\colon X \longrightarrow X_i$ is the i^{th} canonical projection.
(4) Let $x_n = (\underbrace{0, 0, \ldots, 0}_{n \text{ times}}, 1, 1, 1, \ldots)$. Show that the sequence (x_n) does not converge to any point in $\mathbb{R}^\omega_{\text{box}}$ under the box topology.

Solution.

(1) Any basic open set for the product topology is also an open set in the box topology, indicating that the box topology is finer.
(2) Consider the open set $U = \prod_{n\geq 1} \left(-\frac{1}{n}, \frac{1}{n}\right)$ in $\mathbb{R}^\omega_{\text{box}}$. We find that

$$f^{-1}(U) = \left\{x \in \mathbb{R} : -\frac{1}{n} < x < \frac{1}{n}, \forall n \in \mathbb{N}\right\} = \{0\},$$

which is not open in $\mathbb{R}$, thus showing f is not continuous.

(3) If (x_n) converges to x in X, then by the continuity of p_i, $(p_i(x_n))$ converges to $p_i(x)$ for each $i \in I$. Conversely, if for every $i \in I$, $(p_i(x_n))$ converges to $p_i(x)$. Let $U = \prod_{i\in I} O_i$ be an elementary (basic) open set of X_{prod} containing x; with $O_i = X_i$, for $i \in I - J$, J is a finite subset of I, and O_j open in X_j, for all $j \in J$.
For every $j \in J$, the sequence $(p_j(x_n); n \in \mathbb{N})$ converges to $p_j(x)$. So there exists $N_j \in \mathbb{N}$, such that $x_n \in O_j$, for $n \geq N_j$. If we let $N = \max_{j\in J} N_j$, then $x_n \in U$, for every $n \geq \mathbb{N}$. Therefore $(x_n, n \in \mathbb{N})$ converges to x.

(4) Suppose that $(x_n, n \in \mathbb{N})$ converges to an $a = (a_1, a_2, \ldots, a_k, \ldots) \in \mathbb{R}^\omega$ (for the box topology). Take $U = \prod_{i\geq 1} \left(a_i - \frac{1}{i}, a_i + \frac{1}{i}\right) U$, then U is box-open.
So there exists $N \in \mathbb{N}$ such that, $x_n \in U$, for every $n \geq N$.
In particular, we have $a_n - \frac{1}{n} < 0 < a_n + \frac{1}{n}$ and $a_{n+1} - \frac{1}{n+1} < 1 < a_{n-1} + \frac{1}{n+1}$. It follows that $\lim_{n\to+\infty} a_n = 0 = 1$ (on $(\mathbb{R}, \mathfrak{U})$), a contradiction. We conclude that $(x_n, n \in \mathbb{N})$ does not converge to any point of $\mathbb{R}^\omega_{\text{box}}$. However, for each i, the sequence $(p_i(x_n), n \in \mathbb{N})$ converges to zero (it is stationary).

□

Exercise 7. Let X, Y, and Z be topological spaces, and let $f : X \times Y \to Z$ be a function.

(1) Prove that if f is continuous and $(a, b) \in X \times Y$, then the functions defined by

$$\widetilde{f}_a : Y \to Z, \quad y \mapsto f(a, y),$$
$$\widehat{f}_b : X \to Z, \quad x \mapsto f(x, b),$$

are continuous.

(2) Consider the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that the functions $\widetilde{f}_a$ and $\widehat{f}_b$ are continuous for a, b nonzero. When $a = 0$ or $b = 0$, $\widetilde{f}_a$ or $\widehat{f}_b$, respectively, is identically zero. However, f is not continuous at $(0, 0)$.

Proof.

(1) Assuming $f : X \times Y \to Z$ is continuous, let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the canonical projections.
Consider the functions $\eta_a : Y \to X \times Y$, assigning to each $y \in Y$, the pair (a, y), and $\mu_b : X \to X \times Y$, assigning to each $x \in X$, the pair (x, b). Since $p_X \circ \eta_a$ is constant and $p_Y \circ \eta_a = \mathbf{Id}_Y$, we deduce that η_a is continuous. Similarly, since $p_Y \circ \mu_b$ is constant and $p_X \circ \mu_b = \mathbf{Id}_X$, we deduce that μ_b

is continuous. Therefore, the functions $\widetilde{f_a} = f \circ \eta_a$ and $\widehat{f_b} = f \circ \mu_b$ are continuous.

(2) For a, b nonzero, the continuity of $\widetilde{f_a}$ and $\widehat{f_b}$ follows since they are quotients of continuous functions where the denominator is non-zero. When $a = 0$ or $b = 0$, the respective function becomes constantly zero and thus continuous. However, the discontinuity of f at $(0, 0)$ can be demonstrated by considering the sequences $u_n = \left(\frac{1}{n}, \frac{1}{n}\right)$ and $v_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ which both converge to $(0, 0)$, yet $f(u_n) = 1$ and $f(v_n) = -1$, indicating a discontinuity at that point.

$\square$