EXAM III- MATH 453
Duration: 120 mn

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Exercise 1. A subset A of a topological space $(X, \mathcal{T})$ is said to have the fixed point property if every continuous function $f : A \longrightarrow A$ has a fixed point (an $a \in A$ such that $f(a) = a$). Prove that if A has the fixed point property, then A is connected. Is the converse true?

Solution. By contraposition, we will show that if A is not connected, then there exists a continuous function $f : A \to A$ without a fixed point.
Indeed, since A is not connected, there exist two disjoint open sets U, V in A such that $A = U \cup V$. Pick $u \in U$ and $v \in V$, and define f on A by $f(x) = v$ if $x \in U$ and $f(x) = u$ if $x \in V$. Thus, f has no fixed points. Moreover, f is continuous. To see this, it suffices to notice that the preimage of any open set O in A is open. Let O be such an open set. We distinguish four cases:

- If $u \notin O$ and $v \notin O$, then $f^{-1}(O) = \emptyset$, which is open.
- If $u \in O$ and $v \notin O$, then $f^{-1}(O) = V$, which is open.
- If $u \notin O$ and $v \in O$, then $f^{-1}(O) = U$, which is open.
- If $u \in O$ and $v \in O$, then $f^{-1}(O) = U \cup V$, which is open.

The converse is false. It suffices to consider A as the unit circle and f as a rotation about the center 0. □

Exercise 2. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function such that, for every $z \in \mathbb{C}$, we have $f(z)^2 = z^2$.

(1) Consider the function $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $g(z) = \frac{f(z)}{z}$. Use the connectedness of $\mathbb{C} \setminus \{0\}$ to show that $g(z)$ is constant.
(2) Find all such functions f.

Solution. Given any $z \in \mathbb{C}$, the equation $f(z)^2 = z^2$ implies that $f(z)$ could either be z or $-z$. Consider the function $g : \mathbb{C}^* \to \mathbb{C}$ defined by $g(z) = \frac{f(z)}{z}$. The function g is continuous over $\mathbb{C}^*$ and maps into the set $\{-1, 1\}$.
Given that $\mathbb{C}^*$ (the set of all non-zero complex numbers) is path-connected, the image $g(\mathbb{C}^*)$ must also be path-connected. This necessitates that g is constant across its domain. If $g(z) = 1$ for all z, then $f(z) = z$ for every $z \in \mathbb{C}^*$; as in addition $f(0) = 0$, the expression of $f(z)$ extends to all $z \in \mathbb{C}$. Similarly, if $g(z) = -1$, then $f(z) = -z$ for every $z \in \mathbb{C}$. □

Exercise 3. Let X be a connected space and $(O_i)_{i \in I}$ an open covering of X. Show that for any $x, y \in X$, there exists a finite subfamily $(O_{i_p})_{1 \le p \le n}$ such that $x \in O_{i_1}$, $y \in O_{i_n}$, and $O_{i_p} \cap O_{i_{p+1}} \neq \emptyset$ for $1 \le p \le n - 1$.

Solution. For an element $x \in X$, define A_x as the set of points y for which there exists a finite sequence of open sets $(O_{i_p})_{1 \le p \le n}$ connecting x to y as specified. A_x is open because for any point in A_x, there exists an open set O_{i_n} containing it and meeting the criteria for A_x. To show A_x is closed, consider a limit point z

of A_x. There exists an open set O_i containing z that intersects A_x. Extending the sequence to include O_i shows $z \in A_x$, proving A_x is closed. Since A_x is both open and closed in the connected space X, it must be that $A_x = X$, establishing the result. □

Exercise 4. Show that in an unbounded connected metric space, every sphere is non-empty.

Solution. Suppose there exists an empty sphere $S(a,r)$ with $r > 0$. Let $B'(a,r) = \overline{B}(a,r) = \{x \in X : d(a,x) \leq r\}$, then we can express

$$X = B(a,r) \cup (X - B'(a,r)),$$

where both open sets $B(a,r)$ and $X - B'(a,r)$ are non-empty, the latterdue to the unboundedness of X. The connectedness of X then leads to a contradiction. □

Exercise 5. Let A and B be two path-connected subsets of the Euclidean space $E = \mathbb{R}^n$.

(1) Show that $A \times B$ is path-connected.
(2) Deduce that $A + B$ is path-connected.
(3) Is the interior of A always path-connected?

Solution. Let $(a,b) \in A \times B$ and $(a',b') \in A \times B$. Since A is path-connected, there exists a continuous function $f : [0,1] \to A$ such that $f(0) = a$ and $f(1) = a'$. Similarly, since B is path-connected, there exists a continuous function $g : [0,1] \to B$ such that $g(0) = b$ and $g(1) = b'$. Define, for $t \in [0,1]$, $h(t) = (f(t), g(t))$. The function h is continuous, with values in $A \times B$ and satisfies $h(0) = (a,b)$, $h(1) = (a',b')$. Therefore, $A \times B$ is path-connected.
Let $\varphi : A \times B \to E$ be defined by $(a,b) \mapsto a + b$. The function φ is continuous, and $\varphi(A \times B) = A + B$. Since $A \times B$ is path-connected, it follows that $A + B$ is also path-connected.
Consider a counterexample in $\mathbb{R}^2$. Take for A the union of two disjoint balls connected by a segment. This set is path-connected. However, the interior of A, which is the union of the two open balls, is disconnected. So it is not path-connected. □

Exercise 6. Let X be a topological space, $A \subseteq X$, and B be a connected subset of X. Show that if B meets A and $X - A$, then B meets $\mathrm{Fr}(A)$.

Solution. As $X = \mathrm{Int}(A) \cup \mathrm{Fr}(A) \cup \mathrm{Ext}(A)$, we can express B as

$$B = [\mathrm{Int}(A) \cap B] \cup [B \cap \mathrm{Fr}(A)] \cup [\mathrm{Int}(X - A) \cap B].$$

Assume B does not meet $\mathrm{Fr}(A)$, that is $B \cap \mathrm{Fr}(A) = \emptyset$; then $B = (\mathrm{Int}(A) \cap B) \cup (\mathrm{Int}(X - A) \cap B)$ forms a disjoint union.
Since B meets both A and $X - A$, we have $\emptyset \neq B \cap A = \mathrm{Int}(A) \cap B$ and $\emptyset \neq B \cap (X - A) = \mathrm{Ext}(A) \cap B$. Thus, $B = (\mathrm{Int}(A) \cap B) \cup (\mathrm{Int}(X - A) \cap B)$ would be a partition of B into two non-empty open subsets, contradicting the fact that B is connected. Hence, our initial assumption must be wrong, implying $B \cap \mathrm{Fr}(A) \neq \emptyset$. □