FINAL EXAM- MATH 453
Duration: 180 mn

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ID:

Exercise 1. Let $(X, \mathcal{T})$ be a topological space. Show that if X is compact, then every sequence of elements of X has a cluster point (an adherent point).

Solution. The set of cluster points (adherent points) of the sequence $(x_n)_{n\in\mathbb{N}}$ is defined as the intersection of the closures of its tails:

$$\mathrm{Adh}(\{x_n\}) = \bigcap_{n\geq 1} \overline{\{x_k : k \geq n\}}.$$

Denote $C_n = \overline{\{x_k : k \geq n\}}$ for each $n \in \mathbb{N}$. The family $\{C_n\}_{n\in\mathbb{N}}$ consists of a decreasing sequence of nonempty closed subsets of X. Since X is compact, by the finite intersection property (FIP), we have:

$$\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset.$$

Hence, $\mathrm{Adh}(\{x_n\})$ is nonempty, establishing that every sequence in X has at least one cluster point. □

Exercise 2. Let $a < b$ be real numbers. Show that the subspace $K = [a, b]$ of $(\mathbb{R}, \mathfrak{U})$ is both a completion and a compactification of the subspace $I = (a, b)$ of $(\mathbb{R}, \mathfrak{U})$.

Solution. It is evident that K is a compact subset of $(\mathbb{R}, \mathfrak{U})$ since it is a closed and bounded set. Additionally, the map $i\colon (a, b) \to K$ is an embedding, and the closure of $i((a, b))$ is $[a, b] = K$. Therefore, K is a compactification of (a, b).

Furthermore, since every compact metric space is complete, K is complete. Alternatively, K being a closed subset of a complete metric space is also complete. Moreover, the embedding $i\colon (a, b) \to K$ is an isometry, thus K is a completion of (a, b). □

Exercise 3. Let (X, d) be a metric space and $a, b, \alpha, \beta \in X$.

(1) Show that $|d(a, b) - d(\alpha, \beta)| \leq d(a, \alpha) + d(b, \beta)$.

(2) Show that if (x_n) and (y_n) are Cauchy sequences in (X, d), then $(d(x_n, y_n))$ is a Cauchy sequence in $\mathbb{R}$ (equipped with the usual distance).

(3) Let $\widehat{X}$ be the quotient set of the set $\mathbf{Cau}(X)$ of all Cauchy sequences in X by the equivalence relation $\sim$ defined by

$$((x_n) \sim (y_n)) \iff \lim_{n\to+\infty} d(x_n, y_n) = 0.$$

Consider the mapping $\widehat{d} : \widehat{X} \times \widehat{X} \to \mathbb{R}$ defined by

$$\widehat{d}(\widehat{x}, \widehat{y}) = \lim_{n\to+\infty} d(x_n, y_n),$$

where $\widehat{x}$ and $\widehat{y}$ are the equivalence classes of the Cauchy sequences (x_n) and (y_n), respectively.

Show that $\widehat{d}$ is well-defined.

(4) Show that $\widehat{d}$ is a metric on $\widehat{X}$.

Solution. (1) By the triangle inequality, we have

$$d(a,b) \leq d(a,\alpha) + d(\alpha,\beta) + d(\beta,b),$$

and

$$d(\alpha,\beta) \leq d(\alpha,a) + d(a,b) + d(b,\beta).$$

We deduce that

$$-(d(a,\alpha) + d(b,\beta)) \leq d(a,b) - d(\alpha,\beta) \leq d(a,\alpha) + d(b,\beta),$$

showing that $|d(a,b) - d(\alpha,\beta)| \leq d(a,\alpha) + d(b,\beta)$.

(2) Let (x_n) and (y_n) be Cauchy sequences in (X,d) and let $\varepsilon > 0$. Then there exist $p,q \in \mathbb{N}$ such that

$$\begin{array}{lcll} d(x_n,x_m) & \leq & \frac{\varepsilon}{2} & \text{for all } m,n \geq p, \\ d(y_n,y_m) & \leq & \frac{\varepsilon}{2} & \text{for all } m,n \geq q. \end{array}$$

It follows that

$$|d(x_n,y_n) - d(x_m,y_m)| \leq d(x_n,x_m) + d(y_n,y_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

As a result, $(d(x_n,y_n))$ is a Cauchy sequence in $\mathbb{R}$.

(3) By the previous question, $\lim_{n\to+\infty} d(x_n,y_n)$ exists and is nonnegative. Suppose $((x_n) \sim (y_n))$ and $((a_n) \sim (b_n))$. Then

$$\lim_{n\to+\infty} d(x_n,y_n) = 0 = \lim_{n\to+\infty} d(a_n,b_n).$$

On the other hand, by the first question,

$$|d(x_n,a_n) - d(y_n,b_n)| \leq d(x_n,y_n) + d(a_n,b_n).$$

Hence,

$$\lim_{n\to+\infty} |d(x_n,a_n) - d(y_n,b_n)| \leq \lim_{n\to+\infty} d(x_n,y_n) + \lim_{n\to+\infty} d(a_n,b_n) = 0.$$

Therefore, $\lim_{n\to+\infty} d(x_n,a_n) = \lim_{n\to+\infty} d(y_n,b_n)$, meaning that $\widehat{d}$ is well-defined.

(4) Straightforward. □

Exercise 4.

(1) Show that every totally bounded metric space is separable.
(2) Deduce that every compact metric space is second countable.
(3) Use Urysohn Metrization theorem to show that if X is a second countable compact Hausdorff space, then it is metrizable.

Solution. (1) Suppose that (X,d) is totally bounded. Then for each $n \in \mathbb{N}$, there exists a finite subset $A_n \subseteq X$ such that $X = \bigcup_{x\in A_n} B(x, \frac{1}{n})$. Let $A = \bigcup_{n\in\mathbb{N}} A_n$; then A is a countable subset of X. Now for each $x \in X$, we have $d(x, A_n) \leq \frac{1}{n}$ for every $n \in \mathbb{N}$. Thus, $d(x,A) = 0$, and consequently $x \in \overline{A}$, showing that A is dense in X. It follows that X is separable.

(2) Let (X, d) be a compact metric space. By the above proposition, $(X, \mathcal{T}(d))$ is separable. Let A be a dense countable subset of X. Then $\mathcal{B} := \{B(x, \frac{1}{p}) \mid x \in A, p \in \mathbb{N}\}$ is a countable basis for the topology $\mathcal{T}(d)$, showing that $(X, \mathcal{T}(d))$ is second countable.

(3) Urysohn Metrization Theorem (UMT) states that for a space X, the following statements are equivalent.

(i) X is a second countable T_3-space.

(ii) X is separable and metrizable.

Now, assume X is a second countable compact Haussdorff, then as as every compact Haussdorff is a T_3-space, by (UMT), X is metrizable. □

Exercise 5. Let (X, d) be a metric space and let $f\colon (X, d) \longrightarrow (X, d)$ be an injective function. For $x, y \in X$, define $d_f(x, y) = d(f(x), f(y))$.

1. Show that d_f is a metric on X.
2. Show that d and d_f are topologically equivalent if and only if f induces a homeomorphism from (X, d) to $(f(X), d)$ (i.e., f is an embedding).
3. Deduce that if we let $\delta(x, y) = \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|$ for $x, y \in \mathbb{R}$, then δ is a metric topologically equivalent to the usual metric on $\mathbb{R}$.
4. Show that the metric δ defined previously is not complete.

Solution. (1) Straightforward.

(2) Let $f_1\colon (X, d) \longrightarrow (f(X), d)$ be the function induced by f, let $g = \mathbf{1}_X\colon (X, d) \longrightarrow (X, d_f)$ be the identity, and let $h\colon (X, d_f) \longrightarrow (f(X), d)$ be the function induced by f (i.e., $h(x) = f(x)$).

It is clear that h is a bijective isometry, and in particular, it is a homeomorphism. It is also clear that $f_1 = h \circ g$.

- Suppose that d and d_f are topologically equivalent, then g is a homeomorphism, and consequently, f_1 is a homeomorphism, as desired.

- Conversely, assume f_1 is a homeomorphism; then, since $h = f_1 \circ g^{-1}$, we deduce that h is a homeomorphism. As a result, d and d_f are topologically equivalent.

(3) It suffices to show that the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{x}{1+|x|}$ is an embedding. It is clear that f is injective and induces a homeomorphism from $\mathbb{R}$ onto the open interval $(-1, 1)$ (its inverse function assigns to every $x \in (-1, 1)$ the value $\frac{x}{1-|x|}$).

(4) Consider the sequence of real numbers defined by $x_n = n$. Then, for all positive integers n, p, we have:

$$\delta(x_{n+p}, x_n) = \left|\frac{n+p}{1+|n+p|} - \frac{n}{1+|n|}\right| = \frac{p}{(1+n)(1+n+p)} < \frac{1}{1+n} \underset{n\to+\infty}{\longrightarrow} 0.$$

It follows that the sequence (x_n) is Cauchy in $(\mathbb{R}, \delta)$. As $(x_n = n)$ does not converge with respect to the usual metric d_0, it does not converge with respect to δ (since d_0 and δ are topologically equivalent).

Therefore, $(\mathbb{R}, \delta)$ is not a complete metric space. □

Exercise 6. Let X be a topological space. Show that the following statements are equivalent:

(i) The image of any continuous map from X to the real line is a closed and bounded subset of $\mathbb{R}$.
(ii) The image of any continuous map from X to the real line is a bounded subset of $\mathbb{R}$.
(iii) Any continuous map from X to the real line attains its absolute maximum and minimum.

Solution.
$(i) \implies (ii)$: This implication is straightforward as every closed and bounded subset of $\mathbb{R}$ is necessarily bounded.
$(i) \implies (iii)$: If the image of a continuous function to $\mathbb{R}$ is closed and bounded, it must contain its supremum and infimum due to being closed. Therefore, the function attains its absolute maximum and minimum.
$(iii) \implies (ii)$: A function that attains both an absolute maximum and minimum is by definition bounded, hence the image is bounded.
$(ii) \implies (i)$: Assume that for any continuous map $f : X \longrightarrow \mathbb{R}$, the image $f(X)$ is bounded. We need to prove that $f(X)$ is also closed.
Suppose for contradiction that there exists a limit point a of $f(X)$ not in $f(X)$; that is, $a \in \overline{f(X)} \setminus f(X)$.
Define $h : \mathbb{R} \setminus \{a\} \longrightarrow \mathbb{R}$ by $h(t) = \frac{1}{t-a}$ and consider $g = h \circ f$. The function g is continuous on X.
Given a as a limit point, for every positive integer n, there exists $x_n \in X$ with $f(x_n)$ in $(a - \frac{1}{n}, a + \frac{1}{n})$. Hence,

$$|g(x_n)| = \frac{1}{|f(x_n) - a|} > n,$$

indicating that $g(X)$ is unbounded, which contradicts the assumption that images under continuous maps from X to $\mathbb{R}$ are bounded. Thus, $f(X)$ must be closed. □

Exercise 7. Show that every countably compact space is pseudocompact (every continuous function from X to the real line is bounded).

Solution. Suppose X is countably compact and that $f\colon X \longrightarrow \mathbb{R}$ is continuous. The sets

$$U_n = f^{-1}((-n, n)) = \{x \in X : -n < f(x) < n\}$$

form a countable open cover of X, so for some N, the sets $U_1, U_2, \ldots, U_N$ cover X. Since $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_N$, this implies that $U_N = X$. So $-N < f(x) < N$ for all $x \in X$. In other words, f is bounded, so X is pseudocompact. □

Exercise 8. Determine whether the following sets of $(\mathbb{R}^2, \mathfrak{U}_2)$ are compact or not:

(1) $A = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^4 = 1\}$.
(2) $B = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^5 = 2\}$.

Solution.

(1) Given that $x^2 \geq 0$ and $y^4 \geq 0$, the equation $x^2 + y^4 = 1$ ensures $x^2 \leq 1$ and $y^4 \leq 1$. Hence, $|x| \leq 1$ and $|y| \leq 1$, meaning $\|(x, y)\|_\infty \leq 1$. Therefore, A is bounded. Additionally, A is the preimage of $\{1\}$, which is closed, under the continuous mapping $f(x, y) = x^2 + y^4$. Thus, A is also closed, making it a compact subset of $\mathbb{R}^2$.
(2) The set B is unbounded. For any $r > 0$, the point $(r, \sqrt[5]{2 - r^2})$ belongs to B. Note that the fifth root is defined for all real numbers. However, $\|(r, \sqrt[5]{2 - r^2})\|_\infty \geq r$ can be arbitrarily large. Thus, B is not bounded and consequently not compact.

□

Exercise 9. Let $C = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_1 + \cdots + x_n = 1, x_1 \geq 0, \ldots, x_n \geq 0\}$.

(1) Show that C is a compact subset of $\mathbb{R}^n$.
(2) Let $f : C \longrightarrow \mathbb{R}$ be a continuous function such that $f(x) > 0$ for all $x \in C$. Prove that $\inf\limits_{x \in C} f(x) > 0$.

Solution.

(1) Let us show that C is compact. Indeed, C is bounded because, for any $x \in C$, we have $\|x\|_1 = |x_1| + \cdots + |x_n| = x_1 + \cdots + x_n = 1$. Thus, C is bounded. Now, let us show that C is closed. Define $C_0 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_1 + \cdots + x_n = 1\}$ and $C_i = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_i \geq 0\}$ for $i = 1, \ldots, n$. Then $C = C_0 \cap C_1 \cap \cdots \cap C_n$. Each C_i is closed as the preimage of a closed set under a continuous function. Therefore, C is compact.
(2) Since f is continuous on the compact set C, it is bounded and attains its bounds. In particular, there exists $a \in C$ such that $f(a) = \inf_{x \in C} f(x)$. Given that $f(a) > 0$, we conclude that $\inf\limits_{x \in C} f(x) > 0$.

□

Exercise 10.

(1) Show that if (x_n) is a sequence of elements in a topological space $(X, \mathcal{T})$ and $\ell \in X$ is a limit of (x_n), then $K = \{x_n : n \in \mathbb{N}\} \cup \{\ell\}$ is a compact subset of X.
(2) Let $(X, \mathfrak{D})$ be an infinite countable discrete space. Show that the one-point compactification of $(X, \mathfrak{D})$ is homeomorphic to $K = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ (equipped with the subspace topology inherited from $(\mathbb{R}, \mathfrak{U})$).

Solution.

(1) Let $\mathcal{U} = \{U_i : i \in I\}$ be an open covering of K. Choose $i_0 \in I$ such that $\ell \in U_{i_0}$. As (x_n) converges to ℓ, there exists $p \in \mathbb{N}$ such that $x_n \in U_{i_0}$ for every $n \geq p$. Let $i_1, i_2, \ldots, i_p \in I$ be such that $x_j \in U_{i_j}$ for $j = 1, 2, \ldots, p$. Thus, $\{U_{i_j} : j = 0, 1, \ldots, p\}$ is a finite subcover of $\mathcal{U}$. Therefore, K is compact.

(2) Let $\gamma : X \longrightarrow \mathbb{N}$ be a bijection. Then, the function $e : (X, \mathfrak{D}) \longrightarrow (K \setminus \{0\}, \mathfrak{U})$ defined by $e(x) = \frac{1}{\gamma(x)}$ is a homeomorphism. It follows that e extends to a homeomorphism $e^* : X^* \longrightarrow K$.

$\square$