King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH 472
Exam-I
February 12, 2023

NAME:		ID:	

Q		Points
1		23
2		28
3		10
4		23
5		16
Total		100

Q1) Derive a five-point formula to approximate $f^{(4)}(x_0)$ that uses $f(x_0 - 2h)$, $f(x_0 - h), f(x_0), f(x_0 + h)$ and $f(x_0 + 2h)$. Then find the order of the formula.

[Hint: consider the expression $Af(x_0), +Bf(x_0 + h) + Cf(x_0 + 2h) + D\,f(x_0 + 3h)$]

Q2) Consider the nonlinear BVP

$$(y'')^3 = y' + x^2\, y + 1$$
$$y(0) = 1, \qquad y(4) = 2$$

Let $x_0 = 0\,, x_1 = 1, x_2 = 2, x_3 = 3, and\ x_4 = 4$ and $h = 1$.

Recall the centered difference approximations.

$$y'(x_i) = \frac{1}{2h}\big(y(x_{i+1}) - y(x_{i-1})\big) + O(h^2)$$
$$y''(x_i) = \frac{1}{h^2}\big(y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\big) + O(h^2)$$

a) Write the finite difference approximation to the problem, using the notation $w_i \approx y(x_i)$. This is a system of three nonlinear equations in three unknowns.
b) If Newton's method is used to solve the nonlinear system in part (a), then in every iteration a linear system needs to be solved. Write this linear system as $J\,Y = rhs$
c) Is the matrix J in part (b) tridiagonal? diagonally dominant? nonsingular?

Q3) Consider the BVP: $y'' = 2\,y' - y + x\,e^{x} - x,\ \ 0 \le x \le 2, y(0) = 0, y(2) = -4.$
Table (1) provide a comparison between the actual solution and the finite difference solutions (FDM) with two different values of h. Calculate the order of the finite difference method which used in this problem.

x	Actual Solution	FDM with $h_1 = 0.1$	FDM with $h_2 = 0.05$
0.0	0	0	0
0.5	-0.5421	-0.5409	-0.5418
1.0	-1.6409	-1.6398	-1.6406
1.5	-3.2199	-3.2214	-3.2203
2.0	-4	-4	-4

Q4) Consider the linear second-order boundary-value problem

$$y'' = p(x)\, y' + q(x)\, y + r(x) \quad (1)$$

for $a \le x \le b$ with

$$y'(a) = \alpha \quad (2)$$

$$y(b) = \beta \quad (3)$$

Let $x_i = a + ih$ for $i = 0, 1, \cdots, N+1$, where $h = (b-a)/(N+1)$. A finite-difference method results the system of equations:

$$\left(\frac{-w_{i+1} + 2w_i - w_{i-1}}{h^2}\right) = p(x_i)\left(\frac{w_{i+1} - w_{i-1}}{2h}\right) + q(x_i)w_i + r(x_i) \quad i = 0,1,\cdots,N \quad (4)$$

This system involves w_{-1}.

a) Represent the first derivative in equation (2) by the centered difference then find a formula for w_{-1}.

b) Use equation (4) and the formula you obtained in part (a) to assemble the linear system of equations as **A W = rhs**

Q5) Consider the BVP in question 5. Suppose that p, q and r are continuous on $[a, b]$ and $q(x) > 0$ on $[a, b]$. Show that the $(N + 1) \times (N + 1)$ matrix obtained in part (5b) is nonsingular under certain condition.