MATH 503 (Mathematics for Data Science)

T231 – Midterm Exam (75 points)

Name:

ID

Instructions:

1. Mobile phones are not allowed
2. Attempt all questions
3. Each Multiple-Choice question has **5 points**

Multiple Choice (15 points)

Q1. Given the system $\boldsymbol{Ax} = \boldsymbol{b}$.

Which of the following is true about the least squares' solution $\tilde{\boldsymbol{x}}$

a. $\tilde{\boldsymbol{x}}$ is the solution of the problem $\boldsymbol{Ax} = \tilde{\boldsymbol{b}}$, where $\tilde{\boldsymbol{b}}$ is the orthogonal projection of $\boldsymbol{b}$ into the range (column space) of $\boldsymbol{A}$
b. $\tilde{\boldsymbol{x}}$ exists only if $\boldsymbol{b}$ is in the range of $\boldsymbol{A}$
c. $\tilde{\boldsymbol{x}}$ is the solution of the problem $\boldsymbol{Ax} = \tilde{\boldsymbol{b}}$, where $\tilde{\boldsymbol{b}}$ is the orthogonal projection of $\boldsymbol{b}$ into the null space of $\boldsymbol{A}$
d. Least squares do not involve projections

Q2. The vectors $\boldsymbol{x} = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 4 \end{pmatrix}$ and $\boldsymbol{y} = \begin{pmatrix} -2 \\ 0 \\ 2 \\ -2 \end{pmatrix}$ are linearly independent. Define $\boldsymbol{z} = \boldsymbol{x} - 0.5\boldsymbol{y}$.

Let $\mathcal{A} = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$. Which of the following is always **true**?

a. The set $\mathcal{A}$ spans $\mathbb{R}^4$
b. The set $\mathcal{A}$ spans a two-dimensional subspace of $\mathbb{R}^3$
c. The set $\mathcal{A}$ spans a two-dimensional subspace of $\mathbb{R}^4$
d. The set $\mathcal{A}$ spans a three-dimensional subspace of $\mathbb{R}^3$
e. The set $\mathcal{A}$ spans a three-dimensional subspace of $\mathbb{R}^4$

Q3. Suppose $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$ are all linearly independent from each other, where the numbers $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4$ are all real numbers. Which of the following is always true about the linear system?

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

a. The system has a unique solution
b. The system has no solution
c. The system has infinitely many solutions
d. The system has exactly three solutions
e. None of the above

Written Part (60 points)

Q1. (**19 points**) Suppose the matrix below represent a sample from a dataset, with columns corresponding to different data attributes:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 1 & -2 & 0 & 2 \\ 1 & 4 & 3 & 5 \end{pmatrix}$$

a. Find row echelon form for $\boldsymbol{A}$
b. Identify the columns that correspond to redundant data attributes
c. What is the dimension of the vector space spanned by the columns of the matrix (Answer in terms of linear dependence/independence)
d. How many columns in the matrix are necessary for fitting data
$$\boldsymbol{Ax} = \boldsymbol{b}$$
e. How many vectors are needed to span the kernel of the matrix
f. Find a basis for the $\mathrm{Im}(\boldsymbol{A})$ (The range of $\boldsymbol{A}$)

Q2. (**16 points**) Consider the following vectors:

$$\boldsymbol{x}_1 = (1, -1, -2, 2)^T$$

$$\boldsymbol{x}_2 = (2, 1, 2, -1)^T$$

a. Find the $\ell^1, \ell^2, \ell^\infty$ distance between the vectors:

b. Compute the cosine of the angle between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$

Hint: Recall distance between vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is $\|\boldsymbol{x}-\boldsymbol{y}\|$ in any given norm

Q3. (**9 points**) For each of these sets, indicate if it is orthogonal, orthonormal or none (show details of your steps)

$$\mathcal{A}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$\mathcal{A}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$\mathcal{A}_1 = \left\{ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$$

Q4. (**16 points**) Consider fitting the line:

$$d = m_0 + m_1 x$$

to the data

x	-1	0	1	0
d	2	-1	4	1

a. Formulate the linear system to be solved for the unknowns
b. Write the linear system in a matrix form
c. Form the normal equation for the system in (b) above.
d. Solve the normal system in (d) above.

Good Luck!!! 😊