KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

Department of Mathematics

Math 513 First Exam

Semester (211)

Oct. 12, 2021

Name: ....KEY..............................

I.D: ....

Instructions

1. No electronic device (such as programmable calculator, mobile phone, smart watch) is allowed in this exam.
2. Justify your answers. No credit is given for (correct) answers not supported by work.

Question	Points
1	/20
2	/10
3	/10
4	/10
5	/10
Total	/60

Question 1

(20 points)

a. Let $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$ and P is the matrix that diagonalizes A. Find $P^{-1}AP$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 16 = 0$$

$$\Rightarrow 1 - \lambda = \mp 4 \quad \Rightarrow \lambda = 1 \mp 4 \quad \rightarrow \lambda_1 = 5, \ \lambda_2 = -3$$

Hence $P^{-1}AP = D = \begin{vmatrix} 5 & 0 \\ 0 & -3 \end{vmatrix}$ or $D = \begin{vmatrix} -3 & 0 \\ 0 & 5 \end{vmatrix}$

b. Given that

$$\frac{\pi^2 - 3x^2}{12} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos(nx)}{n^2}, \quad -\pi \le x \le \pi.$$

Express $\frac{\pi^2 x - x^3}{12}$ in an infinite series.

$$\int_0^x \frac{\pi^2 - 3t^2}{12} \, dt = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \int_0^x \cos(nt) \, dt$$

$$\left. \frac{\pi^2 t - t^3}{12} \right|_0^x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \left. \sin(nt) \right|_0^x$$

$$\frac{\pi x - x^3}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin(nx)$$

Question 2 (10 points)

Use

$$t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos(nt)}{n^2}, \quad -\pi \le x \le \pi,$$

to expand π^4 in an infinite series.

From Parseval's equality,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} t^4\,dt = \frac{2t^5}{5\pi}\Big|_0^{\pi} = \frac{2}{5}\pi^4$$

$$= \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$= \frac{4\pi^4}{18} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \left(\frac{2}{5} - \frac{2}{9}\right)\pi^4 = 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\pi^4 = 90\sum_{n=1}^{\infty}\frac{1}{n^4}$$

Question 3 (10 points)

Express $f(x) = \frac{x^3}{2}$ in terms of Legendre polynomials.

$$\frac{x^3}{2} = A_0 P_0 + A_1 P_1 + A_2 P_2 + A_3 P_3 ,$$

$$= A_0 + A_1 x + \frac{3}{2} A_2 x^2 - \frac{1}{2} A_2 + \frac{5}{2} A_3 x^3 - \frac{3}{2} A_3 x$$

$$= (A_0 - \frac{1}{2} A_2) + (A_1 - \frac{3}{2} A_3) x + \frac{3}{2} A_2 x^2 + \frac{5}{2} A_3 x^3$$

$\Rightarrow \underline{A_2 = 0}$,

$\frac{5}{2} A_3 = \frac{1}{2} \Rightarrow \underline{A_3 = \frac{1}{5}}$,

$A_0 - \frac{1}{2} A_2 = 0 \Rightarrow \underline{A_0 = 0}$

$A_1 - \frac{3}{2} A_3 = 0 \Rightarrow \underline{A_1 = \frac{3}{10}}$

Thus

$$\frac{x^3}{2} = \frac{3}{10} P_1 + \frac{1}{5} P_3$$

Question 4 (10 points)

The Sturm-Liouville problem $y'' + \lambda y = 0, y'(0) = y'(L) = 0$ has the eigenfunction solutions $y_0(x) = 1$ and $y_n(x) = cos(n\pi x/L)$.
Verify the orthogonality condition.

$$\int_0^L 1 \cdot \cos\left(\frac{n\pi}{L}x\right) dx = \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)\Big|_0^L = 0$$

If $n \neq m$, then

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx$$

$$= \left\{\left(\sin\frac{(n-m)\pi x}{L}\right) \cdot \frac{L}{2(n-m)\pi} + \frac{L}{2(n+m)\pi} \sin\frac{(n+m)x}{L}\right\}\Bigg|_0^L$$

$$= 0 .$$

Question 5 (10 points)

Evaluate: $\int_0^2 x^6 J_3(x)dx$.

Table of Bessel Functions

β	$J_0(β)$	$J_1(β)$	$J_2(β)$	$J_3(β)$	$J_4(β)$	$J_5(β)$	$J_6(β)$	$J_7(β)$	$J_8(β)$	$J_9(β)$	$J_{10}(β)$
0	1	0	0	0	0	0	0	0	0	0	0
2	0.2239	0.5767	0.3528	0.1289	0.0340	0.0070	0.0012	0.0002	0.0000	0.0000	0.0000

$$\int_0^2 \underbrace{x^2}\ \underbrace{x^4 \cdot J_3(x)dx}$$

$$= x^6 J_4(x)\Big|_0^2 - 2\int_0^2 x^5 J_4(x)dx$$

$$= x^6 J_4(x)\Big|_0^2 - 2\left[x^5 J_5(x)\right]_0^2$$

$$= 2^6 J_4(2) - 2\left[2^5 J_5(2)\right]$$

$$= 64(0.034) - 2(32) J_5(2)$$

$$= 64\left[0.034 - 0.007\right]$$

$$= 64(0.027) = 1.728$$