KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

Department of Mathematics

Math 513 Second Exam

Semester (211)

Dec. 5, 2021

Name: ....

KEY

I.D: .... **Ser:**.......

Instructions

1. No electronic device (such as programmable calculator, mobile phone, smart watch) is allowed in this exam.
2. Justify your answers. No credit is given for (correct) answers not supported by work.

Question	Points
1	/20
2	/20
3	/20
Total	/60

Question 1 (20 points)

Use d'Alembert's formula to solve the wave equation, $u_{xx} = 9u_{tt}$, for the initial conditions $u(x,0) = \frac{1}{x^2+1}$, $u_t(x,0) = e^x$ defined for $|x| < \infty$.

Evaluate the displacement at $x = 1$ and $t = \ln 2$.

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau$$

where, $c = \frac{1}{3}$

$f(x) = \frac{1}{x^2+1}$, $g(x) = e^x$

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Thus, $u(x,t) = \frac{1}{2}\left[\frac{1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right] + \frac{3}{2}\int_{x-ct}^{x+ct} e^{\tau}\,d\tau$

$$= \frac{1}{2}\left[\frac{1}{1+x^2+2xct+c^2t^2} + \frac{1}{1+x^2-2xct+c^2t^2}\right]$$

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$$+ \frac{3}{2}\left[e^{x+ct} - e^{x-ct}\right]$$

$$= \frac{1+x^2+c^2t^2}{(1+x^2+c^2t^2)^2 - 4c^2x^2t^2} + \frac{3e^x}{1}\sinh\frac{t}{3}$$

$$= \frac{1+x^2+\frac{1}{9}t^2}{\left(1+x^2+\frac{t^2}{9}\right)^2 - \frac{4}{9}x^2t^2} + 3e^x \sinh\frac{t}{3}$$

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$u(1,\ln 2) \cong 2.414$

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Question 2 (10 points)

The axisymmetric heat equation in an infinitely long cylinder is given by

$$u_t = \frac{1}{r}u_r + u_{rr}, \quad 0 \le r < 5, \ \ t > 0.$$

Assume that we heated this cylinder of radius 5 to the uniform temperature T_0 and then allowed it to cool by having its surface held at the temperature of zero starting from the time $t = 0$. Find $u(r,t)$.

$u(r,t) = R(r)T(t) \Rightarrow \frac{1}{R}\left(R'' + \frac{1}{r}R'\right) = \frac{1}{T}T' = -\frac{k^2}{25}$

$R(r) = J_0\left(\frac{kr}{5}\right)$

$u(5,t) = R(5)T(t) = 0 \Rightarrow R(5) = 0 \Rightarrow J_0(k) = 0$ 5

Solving T: $T_n(t) = A_n e^{-\frac{k_n^2}{25}t}$, $k_n = 1, 2, 3, \dots$

Thus, $u(r,t) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{k_n}{5}r\right) e^{-\frac{k_n^2}{25}t}$ 5

$u(r,0) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{k_n}{5}r\right) = T_0$, thus

$A_n = \frac{2T_0}{25J_1^2(k_n)} \int_0^5 r J_0\left(\frac{k_n}{5}r\right) dr = \frac{2T_0}{k_n J_1(k_n)}$ 8

Thus,

$u(r,t) = 2T_0 \sum_{n=1}^{\infty} \frac{1}{k_n J_1(k_n)} J_0\left(\frac{k_n}{5}r\right) e^{-\frac{k_n^2}{25}t}$ 2

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Question 3 (10 points)

Solve Laplace's equation over the rectangular region

$$0 < x < \pi,\ 0 < y < \pi,$$

with the following boundary conditions:

$$u_y(x,0) = u_y(x,\pi) = 0, \qquad u(0,y) = u(\pi,y) = 1.$$

$u(x,y) = X(x)\,Y(y)$

$X'' - \lambda X = 0 \Rightarrow X_0(x) = \frac{1}{2}A_0 + \frac{1}{2}B_0 x$

$X_n(x) = A_n \cosh(nx) + B_n \sinh(nx)$ 3/

$Y'' + \lambda Y = 0, \quad Y'(0) = Y'(\pi) = 0 \Rightarrow Y_0(y) = 1$

$Y_n(y) = \cos(ny)$ 5/

Therefore,

$$u(x,y) = \frac{1}{2}A_0 + \frac{1}{2}B_0 x + \sum_{n=1}^{\infty}\left[A_n \cosh(nx) + B_n \sinh(nx)\right]\cos(ny)$$

At $x=0$, $u(0,y) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \cos(ny) = 1$

$\Rightarrow A_0 = \frac{2}{\pi}\int_0^{\pi} 1\, dy = 2$ and $A_n = \frac{2}{\pi}\int_0^{\pi} \cos(ny)\,dy = 0$ 5/

Therefore, $u(x,y) = 1 + \frac{1}{2}B_0 x + \sum_{n\geq 1}^{\infty} B_n \sinh(nx)\cos(ny)$

At $x=\pi$, $u(\pi,y) = 1 + \frac{1}{2}B_0\pi + \sum_{n=1}^{\infty} B_n \sinh(n\pi)\cos(ny) = 1$ 5/

$\Rightarrow B_0 = B_n = 0.$

Hence, $u(x,y) = 1.$

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