King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 513 Major Exam 1
The Second Semester of 2021-2022 (212)
Time Allowed: 120 Minutes

Q:1 (15 points) Find matrix exponential of the coefficient matrix of the linear system and solve the linear system using matrix exponential matrix

$$x'(t) = x - 4y + e^{2t}$$
$$y'(t) = x + 5y + e^{t}$$

Eigenvalues of the coefficient matrix are $\lambda = 3, 3$

Sol: Fundamental Set of Solutions $\{e^{3t}, te^{3t}\}$

$$B_t = \begin{bmatrix} e^{3t} & 3e^{3t} \\ te^{3t} & e^{3t}+3te^{3t} \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} \\ te^{3t} \end{bmatrix} = \begin{bmatrix} e^{3t} - 3te^{3t} \\ te^{3t} \end{bmatrix}$$

$$e^{At} = x_1 I + x_2 A = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix} + \begin{bmatrix} x_2 & -4x_2 \\ x_2 & 5x_2 \end{bmatrix}$$

$$= \begin{bmatrix} e^{3t} - 2te^{3t} & -4te^{3t} \\ te^{3t} & e^{3t} + 2te^{3t} \end{bmatrix}$$

$$x_p(t) = e^{At} y(t) = \int_0^t e^{As}\, b(t-s)\, ds, \quad b(t) = \begin{bmatrix} e^{2t} \\ e^{t} \end{bmatrix}$$

$$= \int_0^t \begin{bmatrix} e^{3s} - 2se^{3s} & -4se^{3s} \\ se^{3s} & e^{3s} + 2se^{3s} \end{bmatrix} \begin{bmatrix} e^{2(t-s)} \\ e^{t-s} \end{bmatrix} ds$$

$$= \int_0^t \begin{bmatrix} e^{s+2t} - 2se^{s+2t} - 4se^{2s+t} \\ se^{s+2t} + e^{2s+t} + 2se^{2s+t} \end{bmatrix} ds$$

$$\int s e^{s+2t}\,ds = s e^{s+2t} - e^{s+2t} = (s-1)e^{s+2t}$$

$$\int s e^{2s+t}\,ds = s\frac{e^{2s+t}}{2} - \frac{e^{2s+t}}{4} = \left(\frac{s}{2} - \frac{1}{4}\right)e^{2s+t}$$

$$= \begin{bmatrix} e^{s+2t} - 2(s-1)e^{s+2t} - (2s-1)e^{2s+t} \\ (s-1)e^{s+2t} + \frac{1}{2}e^{2s+t} + \frac{1}{2}(2s-1)e^{2s+t} \end{bmatrix}_0^t$$

$$= \begin{bmatrix} e^{3t} - 2(t-1)e^{3t} - (2t-1)e^{3t} - e^{2t} - 2e^{2t} - e^{t} \\ (t-1)e^{3t} + \frac{1}{2}e^{3t} + \frac{1}{2}(2t-1)e^{3t} + e^{2t} - \frac{1}{2}e^{t} + \frac{1}{2}e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{3t}(1-2t+2-2t+1) - 3e^{2t} - e^{t} \\ e^{3t}\left(t-1+\frac{1}{2}+t-\frac{1}{2}\right) + e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} (4-4t)e^{3t} - 3e^{2t} - e^{t} \\ (2t-1)e^{3t} + e^{2t} \end{bmatrix}$$

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Q:2 (10 points) Find eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 5 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

Sol: $|A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 & 0 \\ 5 & 2-\lambda & 4 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\left((2-\lambda)^2-4\right) + 1\,(10-5\lambda)$

$$= (2-\lambda)\left[\lambda^2 - 4\lambda + \cancel{4} - \cancel{4} + 5\right]$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 5) = 0$$

$$\Rightarrow \quad \lambda = 2, \quad \lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

$\lambda = 2$, $[A-\lambda I \,|\, 0] = \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 5 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Rightarrow x_2 = 0, \quad x_1 = -\frac{4}{5}x_3$

$K_1 = \begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}$

$\lambda = 2-i$, $[A-\lambda I \,|\, 0] = \left[\begin{array}{ccc|c} i & -1 & 0 & 0 \\ 5 & i & 4 & 0 \\ 0 & 1 & i & 0 \end{array}\right]$

$i x_1 = x_2$

$x_2 = -i x_3$

$5x_1 + i x_2 + 4x_3 = 0$

$$x_1 = -i x_2 = -x_3$$

$K_2 = \begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix}, \qquad K_3 = \begin{bmatrix} 1 \\ -i \\ -1 \end{bmatrix}$ for $\lambda = 2+i$

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Q:3 (10 points) Find the Fourier series of $f(t) = \begin{cases} 0 & -\pi \le t \le 0 \\ t & 0 \le t \le \frac{\pi}{2} \\ \pi - t & \frac{\pi}{2} \le t \le \pi \end{cases}$

Sol: $a_0 = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} t\, dt + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} (\pi - t)dt = \frac{\pi}{8} + \frac{1}{\pi}\left(\pi t - \frac{t^2}{2}\right)\Big|_{\frac{\pi}{2}}^{\pi}$

$$= \frac{\pi}{8} + \frac{1}{\pi}\left[\cancel{\pi^2} - \cancel{\frac{\pi^2}{2}} - \cancel{\frac{\pi^2}{2}} + \frac{\pi^2}{8}\right] = \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} t \cos nt\, dt + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} (\pi - t)\cos nt\, dt$$

$$= \frac{1}{\pi}\left[t\frac{\sin nt}{n}\Big|_0^{\frac{\pi}{2}} + \frac{\cos nt}{n^2}\Big|_0^{\frac{\pi}{2}}\right] + \frac{1}{\pi}\left[(\pi - t)\frac{\sin nt}{n}\Big|_{\frac{\pi}{2}}^{\pi} - \frac{\cos nt}{n^2}\Big|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{n\pi}\left[\frac{\pi}{2}\cancel{\sin\frac{n\pi}{2}}\right] + \frac{1}{n^2\pi}\left[\cos\frac{n\pi}{2} - 1\right]$$

$$+ \frac{1}{n\pi}\left[0 - \cancel{\frac{\pi}{2}\sin\frac{n\pi}{2}}\right] - \frac{1}{n^2\pi}\left[(-1)^n - \cos\frac{n\pi}{2}\right]$$

$$= \frac{1}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right] = \frac{1}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

$$b_n = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} t \sin nt\, dt + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} (\pi - t)\sin nt\, dt$$

$$= \frac{1}{\pi}\left[-t\frac{\cos nt}{n} + \frac{\sin nt}{n^2}\right]_0^{\frac{\pi}{2}} - \frac{1}{\pi}\left[(\pi - t)\frac{\cos nt}{n} + \frac{\sin nt}{n^2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \cancel{-\frac{1}{n\pi}\left[\frac{\pi}{2}\cos\frac{n\pi}{2}\right]} + \frac{1}{n^2\pi}\sin\frac{n\pi}{2} - \frac{1}{n\pi}\left[0 - \cancel{\frac{\pi}{2}\cos\frac{n\pi}{2}}\right]$$

$$- \frac{1}{n^2\pi}\left[0 - \sin\frac{n\pi}{2}\right] = \frac{2}{n^2\pi}\sin\frac{n\pi}{2}$$

Q:4 (10 points) Find the complex Fourier series of $f(t) = t^2 - \pi t$ for $-\pi \le t \le \pi$.

Sol: $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}$, $\quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-i\omega_n t}\,dt$

$\omega_n = \frac{n\pi}{\pi} = n$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}(t^2 - \pi t)e^{-int}\,dt$$

$$= \frac{1}{2\pi}\left[(t^2 - \pi t)\frac{e^{-int}}{-in}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(2t-\pi)\frac{e^{-int}}{-in}\,dt\right]$$

$$= \frac{i}{2n\pi}\left[0 - 2\pi^2 e^{-in\pi}\right] + \frac{1}{2n\pi i}\int_{-\pi}^{\pi}(2t-\pi)e^{-int}\,dt$$

$$= \frac{i\pi(-1)^{n+1}}{n} + \frac{1}{2n\pi i}\left[(2t-\pi)\frac{e^{-int}}{-in}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{2}{-in}e^{-int}\,dt\right]$$

$$= \frac{i\pi(-1)^{n+1}}{n} + \frac{1}{2n^2\pi}\left[\pi e^{-in\pi} + 3\pi e^{in\pi}\right] - \frac{1}{n^2\pi}\frac{e^{-int}}{-in}\Big|_{-\pi}^{\pi}$$

$$= \frac{i\pi(-1)^{n+1}}{n} + \frac{2(-1)^n}{n^2} + \frac{1}{n^3\pi}\left[\underbrace{e^{-in\pi} - e^{in\pi}}_{=0}\right]$$

$$= (-1)^n\left[\frac{2}{n^2} - \frac{i\pi}{n}\right], \quad n \neq 0$$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(t^2 - \pi t)\,dt = \frac{1}{2\pi}\left[\frac{t^3}{3} - \pi\frac{t^2}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{6\pi}(2\pi^3) - \frac{1}{4}(\pi^2 - \pi^2) = \frac{\pi^2}{3}$$

$$f(t) = \frac{\pi^2}{3} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}(-1)^n\left[\frac{2}{n^2} - \frac{i\pi}{n}\right]e^{int}$$

Q:5 (15 points) Solve the differential equation $y''-5y'+6y=f(t)$, where $f(t)=\begin{cases} 1 & 0<t<\pi \\ 0 & \pi<t<2\pi \end{cases}$

and $f(t)=f(t+2\pi)$. Use Fourier series $f(t)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{(2n-1)}$.

Sol: To find $y_H(t)$, $y''-y'+\cdot y=0$,

Auxiliary equation $m^2-5m+6=0$, $m=2,3$

$$y_H(t)=Ae^{3t}+Be^{2t}$$

Let $y_p=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(2n-1)t+b_n\sin(2n-1)t$

$$y_p'=\sum_{n=1}^{\infty}-(2n-1)a_n\sin(2n-1)t+(2n-1)b_n\cos(2n-1)t$$

$$y_p''=\sum_{n=1}^{\infty}-(2n-1)^2a_n\cos(2n-1)t-(2n-1)^2b_n\sin(2n-1)t$$

$$y''-5y'+6y=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)t$$

$$\Rightarrow \sum_{n=1}^{\infty}-(2n-1)^2a_n\cos(2n-1)t-(2n-1)^2b_n\sin(2n-1)t$$

$$-5\sum_{n=1}^{\infty}-(2n-1)a_n\sin(2n-1)t+(2n-1)b_n\cos(2n-1)t$$

$$+3a_0+\sum_{n=1}^{\infty}6a_n\cos(2n-1)t+6b_n\sin(2n-1)t$$

$$=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)t$$

$$\Rightarrow 3a_0=\frac{1}{2}\Rightarrow a_0=\frac{1}{6}$$

$$-(2n-1)^2 a_n - 5(2n-1) b_n + 6a_n = 0$$

$$[6-(2n-1)^2] a_n - 5(2n-1) b_n = 0 \rightarrow (1)$$

$$-(2n-1)^2 b_n + 5(2n-1) a_n + 6b_n = \frac{2}{\pi(2n-1)}$$

$$5(2n-1) a_n + [6-(2n-1)^2] b_n = \frac{2}{\pi(2n-1)} \rightarrow (2)$$

Multiply (1) by $(6-(2n-1)^2)$ and (2) by $5(2n-1)$ and add $[6-(2n-1)^2]^2 a_n + 25(2n-1)^2 a_n = \frac{10}{\pi}$

$$a_n = \frac{10}{\pi[(6-(2n-1)^2)^2 + 25(2n-1)^2]}$$

$$b_n = \frac{2[6-(2n-1)^2]}{\pi(2n-1)[(6-(2n-1)^2)^2 + 25(2n-1)^2]}$$

$$y(t) = Ae^t + Be^{2t} + \frac{1}{12} + \sum_{n=1}^{\infty} a_n \cos(2n-1)t + b_n \sin(2n-1)t$$

Q:6 (15 points) Find eigenvalues and eigenfunctions of the boundary value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(\pi) = 0.$$

Find eigenfunction expansion for $f(x) = x$.

Sol: Auxiliary equation is $m^2 + \lambda = 0$.

Case I: $\lambda = 0$, $m = 0, 0$, $y(x) = A + Bx$, $y'(x) = B$

$y'(0) = 0 \Rightarrow B = 0$, $y(\pi) = 0 \Rightarrow A = 0$, Trivial solution

Case II: $\lambda < 0$, let $\lambda = -\alpha^2$, $\alpha > 0$

Then $m^2 - \alpha^2 = 0 \Rightarrow m = \pm\alpha$, $y(x) = A \cosh \alpha x + B \sinh \alpha x$

$y'(x) = A\alpha \sinh \alpha x + B\alpha \cosh \alpha x$

$y'(0) = 0 \Rightarrow B = 0$, $y(\pi) = 0 \Rightarrow A \cosh \alpha\pi = 0 \Rightarrow A = 0$

Trivial Solution.

Case III: $\lambda > 0$, let $\lambda = \alpha^2$, $\alpha > 0$, $m^2 + \alpha^2 = 0$

$y(x) = A \cos \alpha x + B \sin \alpha x$, $y'(x) = -A\alpha \sin \alpha x + B\alpha \cos \alpha x$

$y'(0) = 0 \Rightarrow B = 0$, $y(\pi) = 0 \Rightarrow A \cos \alpha\pi = 0$

Let $A \neq 0$, then $\cos \alpha\pi = 0 \Rightarrow \alpha\pi = \frac{2n-1}{2}\pi$, $n = 1, 2, \ldots$

$$\alpha = \frac{2n-1}{2}, \quad \text{or} \quad \lambda_n = \frac{(2n-1)^2}{4}, \quad n = 1, 2, 3, \ldots$$

$$y_n(x) = \cos\left(\frac{2n-1}{2} x\right), \quad n = 1, 2, 3, \ldots$$

λ_n are the eigenvalues with corresponding eigenfunctions $y_n(x)$. Now to write

$\int^{\pi} \cancel{f(x)\, y_n(x)\, r(x)}\, dx$

$y_n(x)$. Now to write

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \quad c_n = \frac{\int_0^{\pi} f(x) y_n(x) r(x) dx}{\int_0^{\pi} y_n^2(x) r(x) dx}$$

$$\int_0^{\pi} x \cos\frac{2n-1}{2}x \, dx = \frac{2}{2n-1} x \sin\frac{2n-1}{2}x \Big|_0^{\pi} - \int_0^{\pi} \frac{2}{2n-1} \sin\frac{2n-1}{2}x \, dx, \quad r(x) = 1$$

$$= \frac{2\pi}{2n-1} \sin\frac{2n-1}{2}\pi - 0 + \frac{4}{(2n-1)^2} \cos\frac{2n-1}{2}x \Big|_0^{\pi}$$

$$= \frac{2\pi(-1)^{n+1}}{2n-1} - \frac{4}{(2n-1)^2}$$

$$\int_0^{\pi} \cos^2\frac{2n-1}{2}x \, dx = \frac{1}{2}\int_0^{\pi} [1 + \cos(2n-1)x] \, dx = \frac{\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty} \left[\frac{4(-1)^{n+1}}{2n-1} - \frac{8}{\pi(2n-1)^2} \right] \cos\frac{2n-1}{2}x$$