KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

Department of Mathematics and Statistics

Math 513 Final Exam

Semester (231)

DEC. 18, 2023

Name: ....KEY..

I.D: .... Section: ……..

Instructions

1. No electronic device (such as programmable calculator, mobile phone, smart watch) is allowed in this exam.
2. Justify your answers. No credit is given for (correct) answers not supported by work.

Question	Points
1	/15
2	/15
3	/16
4	/12
5	/12
Total	/70

Question 1 (15 points)

Use the separation of variables method to solve the Laplace equation;

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < 1, \qquad 0 < y < 1,$$

subject to the following conditions:

$$u(x,0) = u(0,y) = u(1,y) = 0, \qquad u(x,1) = x.$$

Let $u(x,y) = X(x)\,Y(y) \Rightarrow \dfrac{X''}{X} = \dfrac{-Y''}{Y} = -\lambda$

$X(0) = X(1) = Y(0) = 0$

$X'' + \lambda X = 0 \Rightarrow X_n(x) = \sin(n\pi x)$

5 $Y_n(y) = A_n \cosh(n\pi y) + B_n \sinh n\pi y$

$Y(0) = 0 \Rightarrow A_n = 0$

$$u(x,y) = \sum_1^\infty B_n \sinh(n\pi y)\sin(n\pi x)$$

$$u(x,1) = x = \sum_1^\infty B_n \sinh n\pi \sin n\pi x$$

5 $\Rightarrow B_n \sinh n\pi = 2\int_0^1 x \sin n\pi x \, dx$

$$= 2\left[\frac{1}{n^2\pi^2}\sin n\pi x - \frac{x}{n\pi}\cos n\pi x\right]_0^1$$

$$= \frac{2(-1)^{n+1}}{n\pi}$$

Thus,

5 $$u(x,y) = \frac{2}{\pi}\sum_1^\infty (-1)^{n+1} \frac{\sinh(n\pi y)\sin(n\pi x)}{n \sinh n\pi}$$

Question 2 (9+6 points)

A) Use the fact that $\mathcal{F}\left\{\frac{1}{1+t^2}\right\} = \pi e^{-|\omega|}$ and Parseval's equality to evaluate $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$.

From P.E. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi^2 e^{-2|\omega|} d\omega$

4

$$= \frac{\pi}{2}\left[\int_{-\infty}^{0} e^{2\omega} d\omega + \int_{0}^{\infty} e^{-2\omega} d\omega\right]$$

$$= \frac{\pi}{4}\left[e^{2\omega}\Big|_{-\infty}^{0} - e^{-2\omega}\Big|_{0}^{\infty}\right]$$

5

$$= \frac{\pi}{2}$$

B) Find the Laplace transform of $f(t) = 2[e^{-2t}\sinh 3t + \cos^2 3t]$.

$$f(t) = 2\left[e^{-2t} \cdot \frac{e^{3t} - e^{-3t}}{2} + \frac{1+\cos 6t}{2}\right]$$

$$= e^{t} - e^{-5t} + 1 + \cos 6t$$

$$F(s) = \underbrace{\frac{1}{s-1} - \frac{1}{s+5}}_{3} + \underbrace{\frac{1}{s} + \frac{s}{s^2+36}}_{3}$$

$$= \frac{6}{(s+2)^2-9} + \frac{1}{s} + \frac{s}{s^2+36}$$

Question 3 (10+6 points)

A) Solve the following integral equation: $f(t) = 1 + 2\int_0^t f(t-x)\cos x\,dx$.

$$F(s) = \frac{1}{s} + 2F(s)\cdot\frac{s}{s^2+1}$$

3

$$F(s) = \frac{1}{s} + \frac{2}{(s-1)^2}$$

2

$$f(t) = 1 + 2te^t$$

5

B) Find $\mathcal{L}^{-1}\left\{\frac{3s}{s^2-4s+5}\right\}$.

$$\frac{3s}{s^2-4s+5} = \frac{3(s-2)+6}{(s-2)^2+1} = 3\frac{s-2}{(s-2)^2+1} + 6\cdot\frac{1}{(s-2)^2+1}$$

3

$$\mathcal{L}^{-1}\left\{\frac{3s}{s^2-4s+5}\right\} = 3\cos t\, e^{2t} + 6\sin t\, e^{2t}$$

3

$$= 3e^{2t}[\cos t + 2\sin t]$$

Question 4 (12 points)

Evaluate: $\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx$.

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx = I\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4x+5}\,dx\right)$$

$$= I\left(\oint_C \frac{e^{iz}}{z^2+4z+5}\,dz\right)$$

Upper half plane $\Rightarrow$ a simple pole at $z=-2+i$

$$\oint_C \frac{e^{iz}}{z^2+4z+5}\,dz = 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{z^2+4z+5}; -2+i\right)$$

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$$= 2\pi i \lim_{z\to -2+i} \frac{(z+2-i)e^{iz}}{(z^2+4z+5)}$$

4

$$= 2\pi i \lim_{z\to -2+i} \frac{e^{iz}}{z+2+i}$$

$$= 2\pi i \, \frac{e^{-2i-1}}{2i}$$

4

$$= \frac{\pi e^{-2i}}{e} = \frac{\pi}{e}\left[\cos 2 - i\sin 2\right]$$

Thus, the imaginary Part $= \frac{-\pi}{e}\sin 2$

Question 5 (12 points)

In terms of the error function, use the Laplace transform to solve the heat equation:

$$u_t - u_{xx} = 0, \quad 0 < x < 1, \; t > 0,$$

subject to:

$$u(0,t) = 0, \qquad u(1,t) = \pi, \qquad t > 0,$$

$$u(x,0) = 0, \qquad 0 < x < 1.$$

$$\mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_t\}$$

$$U_{xx}(x,s) = sU(x,s) - u(x,0) \Rightarrow U_{xx} - sU = 0$$

ODE with $U(0,s) = 0$ and $U(1,s) = \frac{\pi}{s}$

$$U(x,s) = A\cosh(x\sqrt{s}) + B\sinh(x\sqrt{s})$$

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Using the BC's $\Rightarrow A = 0$ and $B = \dfrac{\pi}{s\sinh\sqrt{s}}$

Thus $$U(x,s) = \frac{\pi\sinh(x\sqrt{s})}{s\sinh(\sqrt{s})}$$

But $$\frac{\sinh\sqrt{s}\,x}{s\sinh\sqrt{s}} = \frac{e^{(x-1)\sqrt{s}} - e^{-(x+1)\sqrt{s}}}{s(1-e^{-2\sqrt{s}})} = \sum_{n=0}^{\infty}\left[\frac{e^{-(2n+1-x)\sqrt{s}}}{s} - \frac{e^{-(2n+1+x)\sqrt{s}}}{s}\right.$$

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Thus $$u(x,t) = \pi\,\mathcal{L}^{-1}\left\{\frac{\sinh\sqrt{s}\,x}{s\sinh\sqrt{s}}\right\}$$

$$= \pi\sum_{0}^{\infty}\left[\operatorname{erfc}\left(\frac{2n+1-x}{2\sqrt{t}}\right) - \operatorname{erfc}\left(\frac{2n+1+x}{2\sqrt{t}}\right)\right]$$

$$= \pi\sum_{0}^{\infty}\left[\operatorname{erf}\left(\frac{2n+1+x}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{2n+1-x}{2\sqrt{t}}\right)\right]$$

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