KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

Department of Mathematics

Math 513 Midterm Exam

Semester (231)

Oct. 19, 2023

Name: ....KEY................................

I.D: ....**Sec.**.....

Instructions

1. No electronic device (such as programmable calculator, mobile phone, smart watch) is allowed in this exam.
2. Justify your answers. No credit is given for (correct) answers not supported by work.

Question	Points
1	/15
2	/15
3	/10
4	/10
Total	/50

Question 1 (15 points)

Let $A = \begin{pmatrix} 1 & 2 \\ \frac{-1}{2} & 1 \end{pmatrix}$. Find the matrix P that diagonalizes A and the diagonal matrix D such that $P^{-1}AP = D$.

$$\begin{vmatrix} 1-\lambda & 2 \\ -\frac{1}{2} & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 1 \Rightarrow \lambda_1 = 1+i$$

$$\lambda_2 = 1-i$$

Eigen vectors :

For λ_1 : $\begin{pmatrix} -i & 2 \\ -\frac{1}{2} & -i \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -i & 2 \\ 1 & +2i \end{pmatrix} \rightarrow \begin{pmatrix} -i & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow k_2 = r, \ k_1 = -2ik_2$$

Thus, $K_1 = \begin{pmatrix} 2 \\ i \end{pmatrix}$ and $K_2 = \begin{pmatrix} 2 \\ -i \end{pmatrix}$.

$P = \begin{bmatrix} 2 & 2 \\ i & -i \end{bmatrix}$ and $D = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$

$$\begin{bmatrix} -2i & 2i \\ 1 & 1 \end{bmatrix}$$

Question 2 (15 points)

Let $f(x) = x + \pi, \quad -\pi < x < \pi.$

a) Express f in a Fourier series.

b) Use (a) to show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

a) $$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x+\pi)\, dx = 2\pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x+\pi) \cos nx \, dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x+\pi) \sin nx \, dx = \frac{2}{n} (-1)^{n+1}$$

Thus, $$f(x) = \pi + \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx$$

b) Substitute $x = \frac{\pi}{2}$, gives

$$\frac{3\pi}{2} = f\left(\frac{\pi}{2}\right) = \pi + \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin \frac{n\pi}{2}$$

$$= \pi + 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

$\Rightarrow$ $$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Question 3 (10 points)

a) Let $T_n(x)$, $n = 1,2,3, \ldots$ are the eigenfunctions of the Sturm-Liouville problem; $x^3y'' + 2x^2y' + \lambda y = 0$, $y(1) = 5$, $y'(3) = 6$.

Set up the orthogonality condition.

b) Write the Fourier-Legendre expansion of the function $f(x) = 2x^3 - 1$.

(a) $$\frac{d}{dx}\left[x^2 y'\right] + \frac{\lambda}{x} y = 0$$

Thus, $$\int_1^3 \frac{1}{x} T_n(x) \cdot T_m(x)\, dx = 0, \quad n \neq m$$

(b) $$f(x) = 2x^3 - 1 = A P_0 + B P_1 + c P_2 + D P_3 + \cdots$$

$$= A + Bx + \left(\frac{3}{2}x^2 - \frac{1}{2}\right)c + \left(\frac{5}{2}x^3 - \frac{3x}{2}\right) D$$

$$= \left(A - \frac{c}{2}\right) + \left(B - \frac{3}{2}D\right)x + \left(\frac{3}{2}c\right)x^2 + \left(\frac{5}{2}D\right)x^3$$

$\Rightarrow C = 0$, $D = \frac{4}{5}$

$B = \frac{3}{2} D = \frac{3}{2} \cdot \frac{4}{5} = \frac{6}{5}$

$A = -1$

$$f(x) = -P_0 + \frac{6}{5} P_1 + \frac{4}{5} P_3, \quad c_i = 0 \ \forall\, i > 3.$$

Question 4 (10 points)

Assume $J_0(2) = 0.2239$ *and* $J_1(2) = 0.5767$. Evaluate: $\int_0^1 x^3 J_0(2x)dx$.

Let $t = 2x$

$dt = 2\,dx$

$$\int_0^1 x^3 J_0(2x)\,dx = \int_0^2 \frac{t^3}{8} \cdot J_0(t) \cdot \frac{dt}{2}$$

$$= \frac{1}{16}\int_0^2 t^3 J_0(t)\,dt$$

$$= \frac{1}{16}\int_0^2 t^2 \cdot t J_0(t)\,dt$$

$$= \frac{1}{16}\left[t^2 t^1 J_1(t)\Big|_0^2 - 2\int_0^2 t^2 J_1(t)\,dt \right]$$

$$= \frac{1}{16}\left[8J_1(2) - 2\, t^2 J_2(t)\Big|_0^2 \right]$$

$$= \frac{1}{16}\left[8J_1(2) - 8J_2(2) \right] = \frac{1}{2}\left[J_1(2) - J_2(2) \right]$$

Using, $J_0(x) + J_2(x) = \frac{2}{x} J_1(x) \Rightarrow J_0(2) + J_2(2) = J_1(2)$

$\Rightarrow J_2(2) = J_1(2) - J_0(2)$

Hence, $I = \frac{1}{2}\left[J_1(2) - J_1(2) + J_0(2) \right]$

$$= \frac{1}{2}(0.2239) = 0.11195$$