King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 521: General Topology

First Exam, Fall Semester 231 (120 minutes)

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Remark: Solve **6** questions including **Q7**. Show **full details.**

Q1. (16 points)
(a) Find the *maximal* simply ordered subsets of $\mathbb{R}^2$ with the partial ordering

$$(x_0, y_0) \prec (x_1, y_1) \text{ if } x_0 < x_1 \text{ and } y_0 = y_1.$$

(b) Show that $(-1, 1)$ and $\mathbb{R}$ have the same order type.

Q2. (16 points)
(a) Show that $(\mathbb{R} \times \mathbb{R})_{lex} \simeq \mathbb{R}_d \times \mathbb{R}$.
(b) Is $(\mathbb{R} \times \mathbb{R})_{lex}$ metrizable? Clarify.

Q3. (16 Points) Let X be an ordered set considered with the order topology.
(a) Show that if $Y \subset X$ is convex in X, then the order topology on Y is the same as the topology that Y inherits as a subspace of X.
(b) Give an example of a (*non-convex*) subset $Y \subset X$ for which the subspace topology is *strictly* finer than the order topology on Y.

Q4. (16 points)
(a) Show that if $(x_n)_{n\in\mathbb{Z}_+}$ is a sequence for which the image $\{x_n \mid n \in \mathbb{Z}_+\}$ is infinite, then $(x_n)_{n\in\mathbb{Z}_+}$ converges to every point a in $\mathbb{R}$ considered with the finite complement topology.
(b) Prove the *Pasting Lemma*: Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous with $f(x) = g(x)$ for every $x \in A \cap B$. Show that

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$$

is continuous.

Q5. (16 points) Let Y be an ordered set with the order topology and let $f, g : X \to Y$ be continuous.
(a) Show that $\{x \in X \mid f(x) \leq g(x)\}$ is closed in X.
(b) Show that $h(x) = \min\{f(x), g(x)\}$ is continuous (Hint: Use the *Pasting Lemma*).

Q6. (16 points) Let Y be the quotient space obtained from $\mathbb{R}_K$ by collapsing set $K = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ to a point and $\rho : \mathbb{R}_K \to Y$ be the quotient map. Show that
(a) Y is T_1 but not T_2.
(b) $\rho \times \rho : \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ is *not* a the quotient map.

Q7. (20 points) Prove or disprove:

(a) The cartesian product of countable sets is countable.
(b) $f : \mathbb{R} \to \mathbb{R}^w$, $f(t) = (t, t, t, \cdots)$ is continuous, where $\mathbb{R}^w$ has the box topology.
(c) If X is an ordered set with the order topology, then $\overline{(a, b)} = [a, b]$ for any $a, b \in X$.
(d) $\mathbb{R}_K \subsetneqq \mathbb{R}_u$.

Bonus (5 points): Give a metric space (X, d) and a ball of radius r which *strictly contains* some ball of a *strictly bigger* radius $R > r$.

GOOD LUCK