King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 521: General Topology

Second Exam, Fall Semester 231 (120 minutes)

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Remark: Solve **6** questions including **Q7 & Q8**. Show **full details.**

Q1. (16 points) Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of connected spaces, consider $X = \prod\limits_{\alpha\in J} X_\alpha$ with the *product topology* and pick a fixed point $(a_\alpha)_{\alpha\in J} \in X$. For any finite subset K of J, define

$$X_K := \{(x_\alpha)_{\alpha\in J} \mid x_\alpha = a_\alpha \text{ for all } \alpha \notin K\}.$$

Show that

(a) X_K is connected for any any finite subset K of J and $Y := \bigcup\limits_{K \underset{\text{finite}}{\subseteq} J} X_K$ is connected.

(b) $X = \overline{Y}$ and X is connected.

Q2. (16 points) Show that

(a) Every path-connected topological space is connected.

(b) The *topologist's sin function* $\overline{S}$, where

$$S = \{(x, \sin(\frac{1}{x}) \mid \ 0 < x \leq 1\},$$

is connected but *not* path-connected.

Q3. (16 points) Consider the *lower limit topology* $\mathbb{R}_l$.

(a) Show that the set of path-connected components of $\mathbb{R}_l$ is $\mathcal{C} = \{\{a\} \mid a \in \mathbb{R}\}$.

(b) Characterize the continuous maps $f : \mathbb{R} \longrightarrow \mathbb{R}_l$.

Q4. (16 points) Show that

(a) the one point compactification of $\mathbb{R}$ is homeomorphic with S^1.

(b) the one point compactification of $\mathbb{Z}_+$ is homeomorphic with $\{0\}\cup\{\frac{1}{n} \mid n \in \mathbb{Z}_+\} \subset \mathbb{R}$.

Hint: If $f : X \longrightarrow Y$ is a homeomorphism of locally compact Hausdorff topological spaces, then f extends to a homeomorphism of their one-point compactifications.

Q5. (16 points) Let $p : X \longrightarrow Y$ be a *perfect map* (a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for each y in Y). Show that

(a) If X is regular, then Y is regular.

(b) If X is locally compact, the Y is locally compact.

Q6. (16 points) Show that

(a) Every locally compact Hausdorff topological space is regular.

(b) $\mathbb{R}^\omega$ is normal with the uniform topology.

Q7. **(16 points)** Consider $I = [0,1]$. Show that
(a) $I \subset \mathbb{R}$ is compact.
(b) $I \subset \mathbb{R}_c$ is *not* compact.
(c) $I \subset \mathbb{R}_K$ is *not* compact.
(d) $I \subset \mathbb{R}_l$ is *not* limit point compact.

Q8. (20 points) Prove or disprove:

(a) $\mathbb{R}_l$ is metrizable.
(b) A closed subspace of a normal topological space is normal.
(c) Any finite product of Lindelöf topological spaces is Lindelöf.
(d) Every locally compact Hausdorff topological space is completely regular.

GOOD LUCK