King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 521: General Topology

Final Exam, Fall Semester 231 (180 minutes)

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Remark: Solve **7** questions including **Q8 & Q9**. Show **full details.**

Q1. (12 points) Let X be a Hausdorff locally compact topological space and let $Y := X \cup \{\infty\}$ be its one-point compactification.

(a) Show that

$$X \text{ is second countable } \iff Y \text{ is second countable } \iff Y \text{ is metrizable.}$$

(b) Give and example to show that the equivalent properties of X in (a) are not equivalent to "X is metrizable".

Q2. (12 points) A topological space X is *locally metrizable* iff each point x of X has a neighborhood that is metrizable in the subspace topology. [**Hint:** *Urysohn Metrization Theorem:* Every regular second countable space is metrizable.]

(a) Show that a compact Hausdorff locally metrizable space X is metrizable.

(b) Show that a Lindelöf regular locally metrizable space X is metrizable.

Q3. (12 points) Let X be a topological space. Show that

(a) X is a Lindelöf if and only if the collection of *closed* subsets of X has the *countable intersection property.*

(b) X has a metrizable compactification if and only if X is metrizable and second countable.

Q4. (12 points) Let X be a topological space and $\beta(X)$ its Stone-Cech compactification.

(a) Show that if $c(X)$ is an *arbitrary* compactification of X, then there is a *surjective* closed continuous map $g : \beta(X) \longrightarrow c(X)$ that equals the identity on X.

(b) Assume that X is completely regular. Show that

$$X \text{ is connected } \iff \beta(X) \text{ is connected.}$$

Q5. (12 points) Let X be a topological space and $\beta(X)$ its Stone-Cech compactification. Show that

(a) If X is discrete, then $\beta(X)$ is totally disconnected.

(b) If X is completely regular and non-compact, then $\beta(X)$ is not metrizable.

Q6. (12 points) Let (X, d) be a complete metric space. Show that

(a) If $f : X \longrightarrow X$ is a *contraction,* i.e. there is $\alpha < 1$, such that for all $x, y \in X$:

$$d(f(x), f(y)) \leq \alpha d(x, y),$$

then there exists a *unique* point $z \in X$ such that $f(z) = z$.

(b) A subset $S \subset X$ is closed if and only if S (with the subspace topology) is complete.

Q7. (12 points) Let (Y, d) be a metric space, X a topological space and $\mathfrak{F} \subset \mathfrak{C}(X, Y)$.

(a) If $\mathfrak{F}$ is finite, then $\mathfrak{F}$ is equicontinuous.

(b) Suppose $\{f_n\}_{n\in\mathbb{Z}_+}$ is a sequence of equicontinuous functions on a compact subset $K \subset \mathbb{R}$. Show that: if $\{f_n\}_{n\in\mathbb{Z}_+}$ converges *pointwise* on K, then $\{f_n\}_{n\in\mathbb{Z}_+}$ converges *uniformly* on K.

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Q8. (20 points) Prove or disprove:

(a) Every Hausdorff second countable space is metrizable.

(b) Completeness of metric spaces is a topological property.

(c) If X is a completely regular space and Y_1, Y_2 are two compactifications of X having the *extension property*: (every bounded continuous map $f : X \longrightarrow \mathbb{R}$ extends uniquely to a continuous map $g_1 : Y_1 \longrightarrow \mathbb{R}$ and uniquely to a continuous map $g_2 : Y_2 \longrightarrow \mathbb{R}$), then Y_1 and Y_2 are equivalent.

(d) If X is compact, then every sequence $\{f_n : X \longrightarrow \mathbb{R}\}_{n\in\mathbb{Z}_+}$ of pointwise bounded continuous functions has a uniformly convergent subsequence.

Q9. (20 points) In the following table, insert ($\surd$) against each Property **P** that:

(a) passes from a topological space having Property **P** (in the first column) to all of its subspaces, continuous images;

(b) passes from a collection of topological spaces each of which has Property **P** (in the first column) to finite products, countable products, arbitrary products;

(c) is satisfied by all metric spaces;

(d) is satisfied by all compact Hausdorff spaces.

	T_3	$T_{3\frac{1}{2}}$	T_4	Lindelöf	1st countable	2nd countable	metrizable	connected
subspaces								
continuous images								
finite products								
countable products								
arbitrary products								
metric spaces								
compact Hausdorff								

Bonus (5 points): Give a *geometric description* of the Alexandroff (one-point) compactification of the subspace of $X \subset \mathbb{R}$ (with the standard topology) given by

$$X = (0, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4) \cup (4, 5) \cup (5, 6) \cup (6, 7) \cup (7, 8).$$

GOOD LUCK