King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics Sciences

Math 525 - Graph Theory

Semester – 211

Exam II **Dr. M. Z. Abu-Sbeih** **November 9, 2021**

Student No.: ____________. Name: ____

Show all your work. No credits for answers without justification.

Write neatly and eligibly. You may loose points for messy work.

There are 8 problems.

Question	Grade	Out of
1		15
2		15
3		10
4		15
5		15
6		10
7		10
8		10
Total		100

1) (a) State Mengers Theorem for undirected graph.

(b) Let G be a $k \geq 2$ connected graph and let $S = \{v_1, v_{2,} \cdots, v_k\}$ be a set of k vertices. If $u \in V(G) - S$, then there are k paths $u - v_i$ $(i = 1, \cdots, k)$ with only the vertex u in common.

Proof : Let us consider a graph G' such that $V(G') = V(G) \cup \{x\}$ and $E(G') = E(G) \cup \{xy | y \in S\}$. Then G' is k-connected graph. By Menger's theorem there exist k (internal) vertex disjoint $u - x$ path in G'. These path must passes through vertices of S. Since $|S| = k$, every path contain exactly one element from S. Consider corresponding path in G gives required k paths. □

2) (a) State Kuratowski's Theorem.

` A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.

(b) Let G be a plane graph of order n and size m. If the complement $\overline{G}$ is isomorphic to its dual G^*, find n.

Solution:

Assume that G has r faces. Now $|V(G)| = |V(\overline{G})| = |V(G^*)| = n$. Also we have $r = n$. Since

$$|E(\overline{G})| = |E(G^*)|$$

We have $\frac{n(n-1)}{2} - m = m$ or $m = \frac{n(n-1)}{4}$. Substituting in Euler's formula

$2 = n - m + r = n - \frac{n(n-1)}{4} + n = 2n - \frac{n(n-1)}{4}$.

Simplify $n^2 - 9n + 8 = 0$ which implies $(n-1)(n-8) = 0$.

Thus $n = 1$ and $n = 8$.

3) Find the number of regions (faces) r in a maximal outerplanar graph G of order $n \geq 3$.

Solution:

Let the size of G be m. Since the graph is maximal outerplanar, all interior regions are triangles except the outer faces, which is a cycle of length n. Thus

$$3(r-1) + n = 2m$$

Using Euler's formula

$$2 = n - m + r \text{ or } 4 = 2n - 2m + 2r$$

And substituting for $2m$ we get

$$4 = 2n - 2m + 2r = 2n - [3(r-1) + n] + 2r$$

Simplify: $$4 = 2n - 3r + 3 - n + 2r = n - r + 3$$

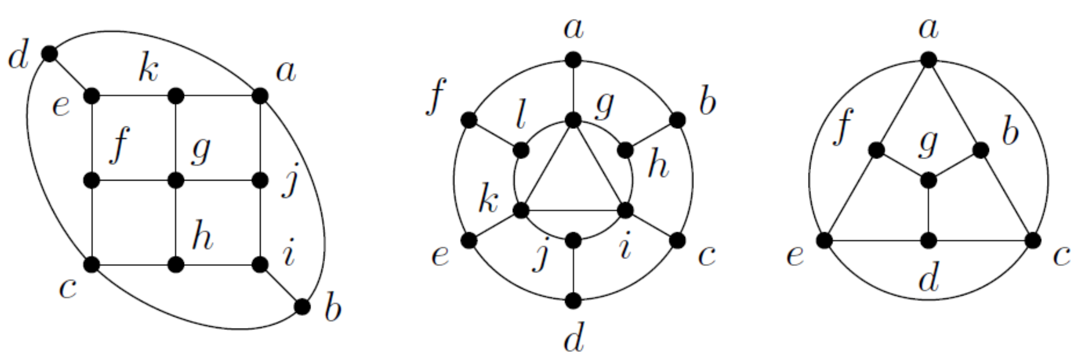
Thus $n - r = 1$ or $r = n - 1$

4) Show that the complement of any planar graph G with at least 11 vertices is nonplanar.

Solution:

A planar graph with n vertices has at most $3n - 6$ edges. Hence each planar graph with 11 vertices has at most 27 edges. Since K_{11} has 55 edges, the complement of each planar subgraph has at least 28 edges and is non-planar. For $n(G) > 11$, any induced subgraph with 11 vertices shows that $\overline{G}$ is nonplanar. There is also no planar graph on 9 or 10 vertices having a planar complement, but the easy counting argument here is not strong enough to prove that.

5) Determine the chromatic number of each of the following graphs:



Solution:

i. The graph has no odd cycles, so $\chi(G) = 2$. (A 2-colouring is easily found (eg., *a, c, e, g, i* white, *b, d, f, h, j, k* black).

ii. Since there are triangles, $\chi(G) \geq 3$. We can find a 3-colouring (eg., *a, d, h, k* red; *b, e, i, l* white; *c, f, g, j* blue), so $\chi(G) = 3$.

iii. Since there are triangles, (eg.,$\{a, b, c\}$), $\chi(G) \geq 3$, but is *G* 3-colourable? If so, without loss of generality let *a, b, c* be red, white, blue respectively. Then *e*, adjacent to both *a* and *c*, must be white. Also *f*, adjacent to both *a* and *e*, must be blue.
Also *d*, adjacent to both *c* and *e*, must be red. Then, however, a fourth color is needed for *g*, which is adjacent to *b, d* and *f*. Hence $\chi(G) = 4$.

6) Show that a simple connected planar graph with 17 edges and 10 vertices cannot be properly colored with two colors.
(Hint: Show that such a graph must contain a triangle.)
Solution:
Suppose such a graph has no triangles. Then, since it is not a tree, each face must be bounded by at least 4 edges, and so $4f \leq 2e$, or $2f \leq e$. However, by Euler's formula, $f = 2 - v + e = 2 - 10 + 17 = 9$, so $2f = 2(9) < 17 = e$. A contradiction. Hence the graph must have at least one triangle and so is not 2-colourable.

7) (a) Show that if *G* is a simple planar graph of order *n* and size *m* then $m \leq 3n - 6$.
(b) Find $cr(K_6)$ (the crossing number of the complete graph on 6 vertices).

Solution:

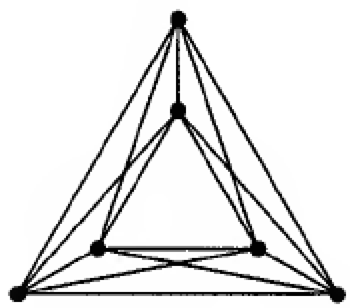
(a) Since each face is bounded by at least 3 edges, if *f* is the no of regions, then $3f \leq 2m$. Substituting in Euler's formula,

$$2 = n - m + f$$

We get $$6 = 3n - 3m + 3f \leq 3n - 3m + 2m = 3n - m$$

That is $m \leq 3n - 6$.

(b) A simple graph on 6 vertices must have at most $3n - 6 = 12$ edges by part (a). But K_6 has 15 edges, then $cr(K_6) \geq 15 - 12 = 3$. The figure bellow shows an embedding of K_6 with three crossings. Hence $cr(K_6) = 3$.



8) Show that every even planar graph *G* is 2-face colorable.
Solution:
We know that if C_1 *and* C_2 are two even cycles then, $C_1 \Delta C_2$ is even. To prove the question, it is enough to show that the dual G^* is two colorable (or bipartite graph). Now since G is even, then each face of the dual is even cycle. However, the faces of the dual form a basis for the cycle space of the dual graph. This implies that any cycle of G^* is the symmetric difference of face cycles, i.e. of even cycles and so is even. Thus, the dual graph is a bipartite graph and can be colored by 2 colors.