MATH 531 - Major Exam 1

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1 Exercise 1(15=4+7+4 points)

1. For any subset $A \subseteq \mathbb{R}$, define $m^*(A)$.
2. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.
3. What does mean to say that a subset $E \subseteq \mathbb{R}$ is measurable.

2 Exercise 2(14=4+10 points)

1. What does mean to say that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is measurable.

2. Prove that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is decreasing (i.e. $f(y) \leq f(x)$ whenever $x \leq y$) then it is measurable

3 Exercise 3(15=8+7 points)

Let $E = \cup_{n=1}^{\infty} \left(n, n + \frac{1}{n^2}\right)$ and $\{f_n\}$ be a sequence of nonnegative integrable functions on E converging uniformly to f on E.

1. Find $m(E)$.
2. Prove that $\lim_{n\to\infty} \int_E f_n = \int_E f$.

4 Exercise 4(15 points)

Let f be a Lebesque integrable function on $\mathbb{R}$. Using the Dominated Convergence Theorem, prove that

$$\lim_{n\to\infty} \int_{[n,n+1]} f = 0.$$

5 Exercise 5(15 points)

Let $\{a_n\}$ be a sequence of nonegative real numbers. Define $f : E = [1, \infty) \longrightarrow \mathbb{R}$ by $f(x) = a_n$ if $n \leq x < n + 1$. Show that $\int_E f = \sum_{n=1}^{\infty} a_n$.

6 Exercise 6(15 points)

Suppose that $f : [0,1] \longrightarrow \mathbb{R}$ is measurable and that there exists $\delta > 0$ such that

$$m\{x \in [0,1] : |f(x)| \leq \frac{1}{n}\} \geq \delta,$$

for any positive integer $n \geq 1$ Show that

$$m\{x \in [0,1] : |f(x)| = 0\} \geq \delta.$$

7 Exercise 7(11=4+7 points)

1. State the Bounded Convergence Theorem
2. Let $\{f_n\}$ be the sequence defined as $f_n(x) = \frac{1}{n}$ if $|x| \leq n$ and 0 otherwise. Find its limit function and prove or disprove if it is satisfies the conclusion of Bounded Convergence Theorem.