MATH 531 - Final Exam

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Instructions: You must show all your work and state all the theorems you use. No materials are allowed.

Exercise 1(8+7 points)

1. Evaluate with proof
$$\lim_{n\to\infty}\int_0^n \left(\frac{\cos\left(\frac{x}{n}\right)}{1+x}\right)^2 dx.$$

2. Evaluate with proof
$$\lim_{n\to\infty}\int_0^n \frac{\cos\left(\frac{x}{n}\right)}{1+x} dx.$$

Exercise 2(10+5+5 points)

1. Let $1 \le p < q < r < \infty$. Show that $L^p(\mathbb{R}) \cap L^r(\mathbb{R}) \subseteq L^q(\mathbb{R})$.

2. Suppose that $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ is Lebesgue measurable for each n such that $\|f_n\|_3 \to 0$ and $\|f_n\|_5 \to 0$ as $n \to \infty$. Prove or give a counterexample for each of the statements below.

   (a) $\|f_n\|_4 \to 0$.

   (b) $\|f_n\|_2 \to 0$.

Exercise 3(5+5 points)

1. Let $f \in L^1(\mathbb{R})$. Show that $\arctan(f) \in L^1(\mathbb{R})$

2. Show that if $f, f_1, f_2, \ldots$ belong to $L^1(\mathbb{R})$ such that $(f_n) \to f$ in $L^1(\mathbb{R})$, then $(\arctan(f_n)) \to \arctan(f)$ in $L^1(\mathbb{R})$.

Exercise 4(7+7 points)

Let $1 < p < \infty$. Suppose that f belongs to $L^p([0, \infty))$.

1. Show that
$$\int_x^\infty \frac{|f(t)|}{t}dt < (p-1)^{\frac{p-1}{p}} x^{-\frac{1}{p}} \left(\int_x^\infty |f(t)|^p dt\right)^{\frac{1}{p}}$$
for any $x > 0$.

2. Deduce that
$$\lim_{x\to\infty} x^{\frac{1}{p}} \int_x^\infty \frac{f(t)}{t}dt = 0.$$

Exercise 5(8+8 points)

1. Let $f : [a,b] \longrightarrow [m,M]$ be an absolutely continuous function and $g : [m,M] \longrightarrow \mathbb{R}$ be Lipchitz. Show that $h = g \circ f$ is absolutely continuous on $[a,b]$.

2. Let f be of bounded variation on $[a,b]$. Show that if $f \geq c$ on $[a,b]$ for some constant $c > 0$, then
$$TV\left(\left(\frac{1}{f}\right)_{[a,b]}\right) \leq \frac{TV(f_{[a,b]})}{c^2}.$$

Exercise 6(6+9 points)

Let $f_n(x) = nx^{n-1} - (n+1)x^n$, for $x \in (0,1)$ and $n \geq 1$.

1. Show that
$$\int_{(0,1)} \sum_{n=1}^\infty f_n \neq \sum_{n=1}^\infty \int_{(0,1)} f_n$$

2. Show that $\sum_{n=1}^\infty \int_{(0,1)} |f_n| = \infty$.

Exercise 7(5+5 points)

Let f be the function defined by $f(x) = x^2 - 4x + 3$. Define ν the signed measure on $\mathbb{R}$ by
$$\nu(E) = \int_E f dm,$$
for any Lebesgue measurable subset E of $\mathbb{R}$.

1. $[2,4]$ is positive? negative? Justify.

2. Find a Hahn decomposition of $\mathbb{R}$ and give the Jordan decomposition of ν. Justify