MATH 531 - Major Exam 1

KFUPM, Department of Mathematics and Statistics

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1 Exercise 1(10 points)

Show that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure.

2 Exercise 2(15 points)

Suppose the function f is defined on a measurable domain E and $\{x \in E : f(x) > c\}$ is a measurable set for each rational number c. f is measurable on E?

3 Exercise 3(15=5+10 points)

1. State Egoroff's Theorem.

2. Show that the conclusion of Egoroff's Theorem can fail if we drop the assumption that the domain has finite measure.

4 Exercise 4(15=5+10 points)

1. State Beppo-Levi's Theorem.
2. Use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ to show

$$\int_0^{\infty} \frac{x}{e^x - 1} dx = \frac{\pi^2}{6}.$$

5 Exercise 5(15=5+10 points)

1. State the Monotone Convergence Theorem.
2. Show that the Monotone Convergence Theorem may not hold for decreasing sequences of measurable functions.

6 Exercise 6(15=5+10 points)

1. State the Lebesgue Dominated Convergence Theorem.

2. Prove that there does not exist an integrable function f on $[0, 1]$ such that for any integer $n \geq 1$,

$$n^2(1-x)x^n \leq f(x) \text{ for all } x \in [0,1].$$

   Hint: Use the fact that if $\lim_{n\to\infty} \left|\frac{f_{n+1}}{f_n}\right| = L < 1$, then $\lim_{n\to\infty} f_n = 0$.

7 Exercise 7(15 points)

Show that the following limit exists and find the limit.

$$\lim_{n\to\infty} \int_{(0,\infty)} \frac{\cos(x^n)}{1+x^n} dm(x).$$