MATH 531 - Major Exam 2

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1 Exercise 1(20 points)

Let f be integrable over $\mathbb{R}$.

1. Show that f is uniformly integrable over $\mathbb{R}$.
2. Let F be the function defined by $F(x) = \int_{-\infty}^{x} f$. Show that F is well-defined and continuous.

2 Exercise 2(15 points)

Assume $m(E) < \infty$. For two measurable functions g and h on E, define

$$\rho(g, h) = \int_E \frac{|g-h|}{1+|g-h|}.$$

Show that $\{f_n\} \to f$ in measure on E if and only if $\lim_{n\to\infty} \rho(f_n, f) = 0$.

3 Exercise 3(10 points)

Let f be of bounded variation on $[a, b]$ such that $|f| \geq c$ on $[a, b]$, where $c > 0$ is constant. Show that $\frac{1}{f}$ is of bounded variation on $[a, b]$.

4 Exercise 4(15 points)

Show that a continuous function on (a, b) is convex if and only if $\varphi\left(\frac{x_1 + x_2}{2}\right) \leq \frac{\varphi(x_1) + \varphi(x_2)}{2}$ for all $x_1, x_2 \in (a, b)$.

5 Exercise 5(15 points)

A real-valued function f defined on $[a, b]$ is said to be K-Lipschitz if

$$|f(y) - f(x)| \leq k|y - x| \text{ for all } x, y \in [a, b].$$

Show that f is K-Lipschitz if and only if i) f is absolutely continuous on $[a, b]$ and ii) $|f'(x)| \leq K$ a.e. on $[a, b]$.

6 Exercise 6(15 points)

Let f be integrable over $\mathbb{R}$. Show that the following assertions are equivalent

1. $f = 0$ a.e. on $\mathbb{R}$.
2. $\displaystyle\int_{\mathbb{R}} fg = 0$ for every bounded measurable function g on $\mathbb{R}$.
3. $\displaystyle\int_{A} f = 0$ for every measurable set A.
4. $\displaystyle\int_{O} f = 0$ for every open set O.

7 Exercise 7(10 points)

Identify which of the following statements is true and which is false. If a statement is true, give reason. If a statement is false, provide a counterexample.

1. Define $f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$
   Then f is of bounded variation on $[0,1]$

2. If $\{f_n\} \longrightarrow f$ pointwise on E, then $\{f_n\} \longrightarrow f$ in measure on E.