MATH 531 - FINAL

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1 Exercise 1(15 points)

Let f be a Lebesgue integrable function on $\mathbb{R}$. Prove that

$$\lim_{n\to\infty}\int_{n^2}^{n^3} f(x)dx = 0.$$

2 Exercise 2(15 points)

Let $p \in (1, \infty)$ and $f, g \in L^p([0,1])$. Show that $|f|^{p-1}|g|$ is integrable on $[0,1]$ and find a bound on its integral.

3 Exercise 3(15 points)

Let $1 \leq p < q < r < \infty$ be three real numbers. Let f be measurable on E such that $f \in L^p(E) \cap L^r(E)$. Show that $\int_E |f|^q < \infty$.

4 Exercise 4(15 points)

Let $\{f_n\}_n$ be a sequence of functions in $L^2([0,1])$ satisfying $\|f_n\|_2 \leq M$ for all $n \geq 1$. In addition, suppose that $\{f_n\}_n \longrightarrow f$ almost everywhere on $[0,1]$. Prove that $f \in L^2([0,1])$ with $\|f\|_2 \leq M$.

5 Exercise 5(15 points)

Let ν be a signed measure on a measurable space $(X, \mathcal{M})$. Define the measure $|\nu| = \nu^+ + \nu^-$. Show that E is null with respect to ν if and only if $|\nu|(E) = 0$.

6 Exercise 6(25 points)

Assume that $m(E) < \infty$. Let $f \in L^\infty(E)$. The goal of this question is to show that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

1. Let $p \geq 1$. Show that $\|f\|_p \leq \|f\|_\infty m(E)^{\frac{1}{p}}$. Deduce that

$$\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty.$$

2. Let $\varepsilon > 0$. Using the fact that $\|f\|_\infty - \varepsilon$ is not an essential upper bound of f, show that there exists $E_\varepsilon \subseteq E$ with $m(E_\varepsilon) > 0$ and $\|f\|_p \geq (\|f\|_\infty - \varepsilon)\, m(E_\varepsilon)^{\frac{1}{p}}$ for all $p \geq 1$.

3. Deduce that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Hint: $\lim_{x\to 0^+} a^x = 1$ if $a > 0$.