جامعة الملك فهد للبترول والمعادن
King Fahd University of Petroleum & Minerals

College of Computing and Mathematics
Department of Mathematics

MATH531 – Real Analysis

Major Exam 2
Term 222

April 11, 2023

Time allowed: 110 Minutes.

Name	
Student ID #	

Question #	Mark	Maximum Mark
1		15
2		10
3		23
4		11
5		16
6		13
7		12
Total		100

Instruction: Give the details of every solution (proof).

Exercise 1 [15 *points*]

(a) State the Monotone Convergence Theorem.

(b) Show that Fatou's Lemma implies the Monotone Convergence Theorem.

(c) Let $\{a_n\}$ be a sequence of nonnegative real numbers. Define the function f on $E = [1, \infty)$ by setting $f(x) = a_n$ if $n \leq x < n + 1$. Show that $\int_E f = \sum_{n=1}^{\infty} a_n$.

Exercise 2 [10 *points*]

Let $\{f_n\}$ be a sequence of integrable functions on E for which $f_n \to f$ a.e. on E and f is integrable over E. Show that $\int_E |f - f_n| \to 0$ if and only if $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$.

Exercise 3 [23 *points*]

(a) Let $\mathcal{F}$ be a family of functions and E be a measurable set. What does it mean to say that $\mathcal{F}$ is uniformly integrable over E and that $\mathcal{F}$ is tight over E?

(b) Let $\mathcal{F}$ be a family of functions, each of which is integrable over E. Show that $\mathcal{F}$ is uniformly integrable over E if and only if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $f \in \mathcal{F}$, if U is open and $m(E \cap U) < \delta$, then $\int_{E\cap U} |f| < \varepsilon$.

(c) Let $\{f_k\}_{k=1}^{n}$ be a finite family of functions, each of which is integrable over E. Show that $\{f_k\}_{k=1}^{n}$ is tight over E.

Exercise 4 [11 *points*]

(a) Show that if $\{f_n\} \to f$ uniformly on E, then $\{f_n\} \to f$ in measure on E

(b) Let $\{f_n\}$ be a sequence of nonnegative integrable functions on E. Show that if $\lim_{n\to\infty} \int_E f_n = 0$ then $\{f_n\} \to 0$ in measure on E.

Exercise 5 [16 *points*]

(a) Let f be a bounded function on $[a, b]$ whose set of discontinuities has measure zero. Show that f is measurable. Then show that the same holds without the assumption of boundedness.

(b) Show that if f is defined on (a, b) and $c \in (a, b)$ is a local minimizer for f, then $\underline{D}f(c) \leq 0 \leq \overline{D}f(c)$.

Exercise 6 [13 *points*]

(a) [***Continuity of Integration***] Let f be integrable over E and $\{E_n\}_{n=1}^{\infty}$ be descending countable collection of measurable subsets of E. Show that

$$\int_{\bigcap_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f.$$

(b) Let f be an integrable function on E. Show that for every $\varepsilon > 0$ there exists a natural number N such that if $n \geq N$, then $\left|\int_{E_n} f\right| < \varepsilon$ where $E_n = \{x \in E \mid |x| \geq n\}$.

Exercise 7 [12 *points*]

(a) Compute the total variation (TV) of e^{2x} on the interval $[0, 10]$.

(b) Let the function f be of bounded variation on the closed, bounded interval $[a, b]$. Show that $f(x) + TV(f_{[a,x]})$ and $TV(f_{[a,x]})$ are increasing on $[a, b]$ for all $x \in [a, b]$.