جامعة الملك فهد للبترول والمعادن
King Fahd University of Petroleum & Minerals

College of Computing and Mathematics
Department of Mathematics

MATH531 – Real Analysis

Final Exam - Term 222

May 29, 2023

Time allowed: 150 Minutes.

Name	
Student ID #	

Question #	Mark	Maximum Mark
1		20
2		20
3		15
4		30
5		20
6		10
7		25
Total		140

Instruction: Give the details of every solution (proof).

Exercise 1 (*20 points*)

(a) Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets. Prove that for any set A,

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k)$$

(b) Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f and f is finite a.e. on E. Show that $\{f_n\} \to f$ in measure on E.

Exercise 2 (*20 Points*)

(a) Let the function f be of bounded variation on the closed, bounded interval $[a, b]$. Show that f is differentiable almost everywhere on the open interval (a, b).

(b) Let the function f be absolutely continuous on $[a, b]$. Show that f is Lipschitz on $[a, b]$ if and only if there is a $c > 0$ for which $|f'| \leq c$ a.e. on $[a, b]$.

Exercise 3 (*15 Points*)

(a) Let f have a second derivative at each point in (a, b). Show that f is convex if and only if f'' is nonnegative.

(b) Let f be integrable over [0,1]. Show that

$$\exp\left[\int_0^1 f(x)dx\right] \leq \int_0^1 \exp\big(f(x)\big)\, dx.$$

Exercise 4 (*30 Points*) True or False. If the statement is true, prove it. If the statement is false, provide a counterexample (with proof).

(a) Every Cauchy sequence of real numbers is rapidly Cauchy.

(b) If $\{f_n\}$ is bounded in $L^1[0,1]$, then $\{f_n\}$ uniformly integrable over [0,1].

(c) Let $1 < p < \infty$. If $f, g \in L^p([0,1])$, then $|f||g|^{p-1}$ is integrable on [0,1].

Exercise 5 (*20 Points*)

(a) For f in $L^1[a, b]$, define $\|f\| = \int_a^b x^2|f(x)|$. Show that this is a norm in $L^1[a, b]$.

(b) Let $\{f_n\}$ be a sequence of measurable functions that converges pointwise a.e. on E to f. Suppose there is a function g in $L^2(E)$ such that for all n, $|f_n| \leq g$ a.e. on E. Show that $\{f_n\} \to f$ in $L^2(E)$.

Exercise 6 (*10 Points*)

Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Show that the union of a countable collection of positive sets is positive.

Exercise 7 (*25 Points*)

(a) Let f and g be nonnegative measurable functions on X for which $g \leq f$ a.e. on X. Show that $f = g$ a.e. on X if and only if $\int_X g d\mu = \int_X f d\mu$

(b) Let $(X, \mathcal{M}, \mu)$ be a measure space and f a nonnegative measurable function on X for which $\int_X f d\mu < \infty$. Show that f is finite a.e. on X and $\{x \in X | f(x) > 0\}$ is σ-finite.