King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH533 - Complex Variables
Exam I – Semester 231

Exercise 1

Let f be an entire function.

(a) Show that if $\lim\limits_{|z|\to\infty} f(z) = c \in \mathbb{C}$, then f is constant

(b) Let $a \in \mathbb{C}$, $r > 0$. Show that if $f : \mathbb{C} \to \mathbb{C} \setminus D(a, r)$, then f is constant.

Exercise 2

Under the transformation $w = \dfrac{iz}{z-1}$ find the image of

(a) the open unit disk $|z| < 1$.

(b) the right half-plane $\operatorname{Re} z > 0$.

Exercise 3

(a) Show that

$$\cot^{-1}(z) = \frac{1}{2i} \log\left(\frac{z+i}{z-i}\right)$$

and find its derivative.

(b) Find the domain of analyticity of

$$\mathrm{Cot}^{-1}(z) := \frac{1}{2i}\mathrm{Log}\left(\frac{z+i}{z-i}\right)$$

Exercise 4

Let $w = T(z) = \dfrac{az+b}{cz+d}$ be a bilinear transformation such that $ad - bc \neq 0$ and $T(z_1) = w_1$ and $T(z_2) = w_2$ with $z_1 \neq z_2$.

(a) Show that

$$w - w_k = \frac{(ad-bc)(z-z_k)}{(cz+d)(cz_k+d)}. \quad k = 1, 2.$$

(b) Prove that

$$\frac{w-w_1}{w-w_2} = \left(\frac{cz_2+d}{cz_1+d}\right)\left(\frac{z-z_1}{z-z_2}\right).$$

(c) Deduce that, if z_1 and z_2 are distinct fixed points of a bilinear transformation $w = T(z)$, show that

$$\frac{w-z_1}{w-z_2} = K\frac{z-z_1}{z-z_2}$$

where $K \in \mathbb{C}$.

(d) Using (c), find the bilinear transformation T having z_1 and z_2 as distinct fixed points and $T(\infty) = \dfrac{z_1+z_2}{2}$.

Exercise 5

Let u be a harmonic function on a connected domain Ω.

(a) Show that $g(z) = u_x - iu_y$ is analytic on Ω.

(b) Show that if G is an antiderivative of g, then $\operatorname{Re} G - u$ is constant.

(c) Show that u has a harmonic conjugate if and only if g has an antiderivative on Ω.

(d) Use (c) to show that the harmonic function $z \mapsto \ln|z|$ has no harmonic conjugates on $\mathbb{C} \setminus \{0\}$.

Exercise 6

Find all entire functions f such that

$$|f(z)| \leq \frac{|z|^3}{\log |z|} \quad |z| > 1.$$

Exercise 7

(a) Evaluate the following integral

$$\int_{|z|=1} z(\operatorname{Re} z)^2 \, dz$$

(b) Suppose that f is analytic for $|z| < 2$. Evaluate

$$\int_{|z|=1} (\operatorname{Im} z + 1) \frac{f(z)}{z} dz$$