Department of Mathematics, King Fahd University of Petroleum & Minerals,
Math 533 Exam-01, 2024-2025 (241)

Time Allowed: 120 Minutes

Name: ______________________ ID#: ______________

Section/Instructor: ______________ Serial #: ______________

- Mobiles, calculators and smart devices are not allowed in this exam.
- Write neatly and eligibly. You may lose points for messy work.
- Show all your work. **No points for answers without justification.**

Question #	Marks	Maximum Marks
1		17
2		16
3		8
4		7
5		12
Total		60

Q1: **(a)** Show that

$$(i) \qquad \frac{R^4 - R}{R^2 + R + 1} \leq \left| \frac{z^4 + iz}{z^2 + z + 1} \right| \leq \frac{R^4 + R}{(R-1)^2} \ \textit{for all } z \textit{ satisfying } |z| = R > 1.$$

$$(ii) \qquad |1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2).$$

(b) Express $\left(\dfrac{1+i}{\sqrt{2}}\right)^{1337}$ in the form $x + iy, x, y \in \mathbb{R}$

(c) Find the four roots of the polynomial $z^4 + 16$ and use these to factor $z^4 + 16$ into two quadratic polinomials with real coefficients.

Q2: **(a)** Let $f(x) = \begin{cases} \dfrac{x^2y^5(x+iy)}{x^4+y^{10}}, & z \neq 0 \\ \\ 0, & z = 0. \end{cases}$.

(i) Show that the Cauchy-Riemann equations hold at origin.

(ii) Determine whether or not $f(z)$ is differentiable at origin.

(b) If f is a differential function at z, show that $|f'(z)|^2 = J(u,v)$ (Jacobian of u and v with respect to x and y.
(c) Let $u(x,y) = x^4 + y^4 - 6x^2y^2 - 4xy$. Determine entire function $f(z)$ for which $f(0) = i$.

Q3: (a) Let $|f(z)|$ be constant in a domain D, where $f(z)$ is analytic. Then show that $f(z)$ is constant in D.

(b) Let $f(z)$ be a differentiable function. Show that $\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} = df$.

Q4: (a) For what values of z is $\sum_{n=0}^{\infty}\left(\frac{z}{1+z}\right)^n$ convergent?

OR

(a) Discuss the convergence of $\sum_{n=-\infty}^{\infty} 3^{-|n|}\, z^{2n}$.

(b) Evaluate $\int_{|z|=r} x dz$. OR $\int_{|z|=1} |z-1||dz|$.

Q5: (a) Evaluate $\int_{\pi}^{\pi+2i} \cos\frac{\mathrm{z}}{2}\, dz$ by using Fundamental theorem of contour integral.
(b) State and prove ML-inequality and use it to show that

$$\lim_{n\to\infty} \int_{c_n} \frac{1}{z^4\ \mathrm{cosz}} dz = 0,$$

where $c_n = \{(x, y) : x = \pm n\pi, y = \pm n\pi\}, n = 1, 2, \cdots$.