Department of Mathematics, King Fahd University of Petroleum & Minerals,
Math 533 Final Exam, 2024-2025 (241)

Q1: (a) If u and e^u are harmonic on a domain D, then show that u is identically constant.
(b) Let $f(z)$ be analytic function in a domain containing $|z-z_0| \leq R$ and let u be a harmonic function in a domain containing $|z-z_0| \leq R$. Show that

$$u(z_0) = \frac{1}{\pi R^2} \int\int_{|z-z_0|\leq R} u(x,y)dxdy.$$

Q2: (a) State **Casorati- Weierstrass theorem**. Show that $z = \pi$ is a removable singularity of $f(z) = \dfrac{1}{\sin(\mathrm{z})} - \dfrac{1}{z} + \dfrac{2z}{z^2-\pi^2}$.
(b) If z_0 is a pole of a function $f(z)$, then show that $\lim_{z\to z_0} f(z) = \infty$.

Q3: (a) Evaluate $\int_0^\pi \dfrac{d\theta}{(a+\cos(\theta))^2}$ $a > 1$.
(b) Find the residue of $f(z) = (z^2+a^2)^{-n}, a > 0, n > 1$ at $z = ia$.

OR

(b) Find the residue of $f(z) = \dfrac{\cot(\pi \mathrm{z})}{z^2}$ at $z = 0$.

Q4: State and Prove **Rouche's theorem.** State the **Argument principle** and use it to compute the value of the integral

$$\int_C \frac{e^z}{e^z-2}dz,$$

where C is the positively oriented boundary of $R = \{z = x+iy, 0 < x < 1, -9\pi < y < 9\pi\}$.

Q5: (a) **State Schwarz Lemma.** Let f be analytic on a closed disk of radius r and center $a \in \mathbb{C}$. Let $M = \max_{z\in\bar{B}(a,r)} |f(z)|$.
Prove that

$$z \in \bar{B}(a, r/2), z \neq a,, \frac{f(z)-f(a)}{|z-a|} \leq \frac{2M}{r}.$$

(b) If $f(z)$ is analytic and $Imf(z) \geq 0$ for $Im(z) > 0$, show that

$$\left|\frac{f(z)-f(z_0)}{f(z)-\overline{f(z_0)}}\right| \leq \left|\frac{z-z_0}{z-\bar{z}_0}\right|$$

and

$$\frac{|f'(z)|}{Imf(z)} \leq \frac{1}{y}.$$

Q6: State **Mittag-Leffler's expansion theorem** and use it to expand $f(z) = \cot(\pi \mathrm{z}), \mathrm{z} \neq 0$. Also, show that

$$\frac{\pi^2}{\cos(\pi z)^2} = \sum_{-\infty}^{\infty} \frac{1}{(z-n-\frac{1}{2})^2}.$$