Department of Mathematics, King Fahd University of Petroleum & Minerals,
Math 533 Final Exam, 2025-2026 (251)
Time Allowed: 160 Minutes

Name: ______________________ ID#: ______________

Section/Instructor: ______________ Serial #: ______________

- Mobiles, calculators and smart devices are not allowed in this exam.
- Write neatly and eligibly. You may lose points for messy work.
- Show all your work. **No points for answers without justification.**

Question #	Marks	Maximum Marks
1		(4+8) + 8 =20
2		10+10+10=30
3		10+10+10=30
4		12 + 6 + 12= 30
5		12+12+6=30
Total		140

Q1:(a) (i) Find the image of the right half-plane $\Re(z) > 1$ under the linear transformation

$$w = (-1+i)z + 2 + 3i.$$

(ii) Find the image of the circle $|z-2| = 1$ under the inverse mapping $w = \frac{1}{z}$.

(b) Show that the Möbius transformation

$$w = \frac{z-1}{z+1}$$

maps the right half–plane $\{z \in \mathbb{C} : \Re z > 0\}$ onto the open unit disk $\{w \in \mathbb{C} : \ |w| < 1\}$.

Q2:(a) Identify and classify the singularities of the following functions

$$(i) \;\; g(z) = \frac{e^{1/z}}{z-2}, \qquad (ii) \quad h(z) = \frac{\sin z}{z^4},$$

(b) Suppose f is analytic on an open set D with an isolated singularity at z_0.
(i) Suppose z_0 is a pole of order m. Show that $|f(z)| \to \infty$ as $z \to z_0$.

(ii) Let $|f(z)| \to \infty$ as $z \to z_0$. Show that f has a pole of order m at z_0.

(c) If $f(z)$ has an isolated singularity at $z = a$ and $|f(z)|$ is bounded in the deleted neighbourhood of a, then show that a is a removable singularity.

Q3(a) Evaluate

$$I = \int_0^{2\pi} \frac{1}{a + \sin\theta}\, d\theta$$

where a is real > 1.
(b) Find the residue of

$$f(z) = \frac{1}{z^n - 1}$$

at $z = 1$, where $n \in \mathbb{N}$, $n \geq 2$.

(c) Evaluate

$$\oint_{|z|=2} (2z-1)\, e^{\frac{z}{z-1}}\, dz \qquad \text{(Hint: use the residue at } \infty\text{).}$$

Q4: [(a)] Let f be analytic inside and on a simple closed contour C except for a finite number of poles inside C. Denote the zeros of f by $z_1, \ldots, z_n$ (none lies on C) and the poles by $w_1, \ldots, w_m$, each listed with its multiplicity/order. If g is analytic inside and on C, prove that

$$\frac{1}{2\pi i} \int_C g(z) \, \frac{f'(z)}{f(z)} \, dz = \sum_{i=1}^{n} m_i \, g(z_i) - \sum_{j=1}^{m} n_j \, g(w_j),$$

where m_i and n_j are the multiplicities of the zeros and the orders of the poles, respectively.

(b) Use Argument Principle to evaluate $\oint_C \frac{z+i}{z^2+2iz-4}\,dz$, where $C = \{z : |z+1+i| = 2\}$.

(c) State the Rouche$'s$ theorem. Let λ be real and $\lambda > 1$. Show that the equation

$$ze^{\lambda - z} = 1$$

has exactly one solution in the disc $|z| = 1$, which is real and positive.

Q5: (a) State and prove Schwarz's Lemma.

(b) Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Suppose $|f(z)| \leq 1$ for $|z| < 1$ and f is a non-constant analytic function. By considering the function $g : D \to D$ defined by

$$g(z) = \frac{f(z) - a}{1 - \overline{a} f(z)}$$

where $a = f(0)$, prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)|\,|z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)|\,|z|} \qquad \text{for } |z| < 1.$$

(c) Does there exist an analytic function $f : D \to D$ with

$$f\left(\tfrac{1}{2}\right) = \tfrac{3}{4} \quad \text{and} \quad f'\left(\tfrac{1}{2}\right) = \tfrac{2}{3}\,?$$

(Justify your answer)

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