King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 535: Functional Analysis Exam 1, Spring Semester 212

Problem 1: (20 points)

Let V be a finite-dimensional vector space and let U_1 and U_2 be two sub- spaces of V. Show that one can write

$$\dim\,(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim\,(U_1 \cap U_2).$$

Problem 2:(20 points)

Give an example of a complete metric space and another example for a metric space that is not complete. Justify your answer.

Problem 3: (30 points)

Let X be a metric space with a dense subset $A \subset X$ such that every Cauchy sequence in A converges in X. Prove that X is complete.

Problem 4: (30 points)

Let (X, d) be a complete metric space. Let $\{f_n : X \to X, n = 1, 2, \cdots\}$ be a sequence of contractions with the same constant k and let $\{f : X \to X\}$ be a contraction with the same constant k such that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in X$. If the fixed point of f is x^* and the fixed point of f_n is x_n^* for $n = 1, 2, \cdots$, then show that

$$x^* = \lim_{n\to\infty} x_n^*.$$

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Good luck

Manal Alotaibi

Problem 1: (20 points)

Let V be a finite-dimensional vector space and let U_1 and U_2 be two sub- spaces of V. Show that one can write

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Le V be init 'mensiona vec-or space U_1, U_2 are spaces of V
by a theorm $U_1 \cap U_2$ has finite basis $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$

Part of a basis $\{\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_m\}$ for U_1 $\Rightarrow \dim U_1 = k+m$

and part of a basis $\{\alpha_1, \alpha_2, \dots, \alpha_k, \gamma_1, \gamma_2, \dots, \gamma_n\}$ for U_2 $\Rightarrow * \Rightarrow \dim U_2 = k+n$

Then the subspace $U_1 + U_2$ is spanned by the vectors

$$\{\alpha_1, \alpha_2, \alpha_3 \dots \alpha_k, \beta_1, \beta_2, \dots \beta_m, \gamma_1, \gamma_2, \dots, \gamma_n\}$$

and these vectors form an Independent set. i.e.

$$\sum x_i \alpha_i + \sum y_j \beta_j + \sum z_r \gamma_r = 0$$

$$\Rightarrow \sum x_i \alpha_i + \sum y_j \beta_j = -\sum z_r \gamma_r$$

$\Rightarrow \Rightarrow \sum z_r \gamma_r$ belonge to U_1. As it belonge to U_2 as given In $*$

$$\sum z_r \gamma_r = \sum e_i \alpha_i$$

for certain Scalars $e_1, c_2, \dots c_k$ becouse of $*$ is Independent $z_r = 0 \ \forall r$

$$\Rightarrow \sum x_i \alpha_i + \sum y_j \beta_j = 0$$

and since $\{\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_m\}$ is also Independent

$$\Rightarrow x_i = 0 \text{ and } y_j = 0 \quad \forall i, j$$

Thus, $\{\alpha_1, \alpha_2, \alpha_3 \dots \alpha_k, \beta_1, \beta_2, \dots \beta_m, \gamma_1, \gamma_2, \dots, \gamma_n\}$
is a basis for $U_1 + U_2$ and has the dimention $k+m+n$

$$\begin{aligned} \Rightarrow \dim U_1 + \dim U_2 &= (k+m) + (k+n) \\ &= k + (m+k+n) \\ &= \dim(U_1 \cap U_2) + \dim(U_1 + U_2) \end{aligned}$$

Therefore $\boxed{\dim(U_1 \cap U_2) = \dim U_1 + \dim U_2 - \dim(U_1 + U_2)}$

Problem 2:(20 points)

Give an example of a complete metric space and another example for a metric space that is not complete. Justify your answer.

* Example of a complete metric space.

The metric space $(\mathbb{R}, d)$ is complete

where $d(x,y) = |x-y| \quad \forall\, x,y \in \mathbb{R}$

By considring cauchy seq uence $\{X_n\}$ in $(\mathbb{R}, d)$

- we can immediately get $\{X_n\}$ is bounded by theorem (5.4.3.a)
- But we know that every bounded sequence of real numbers
- must have convergent subsequence By theorem (3.4.2)
- thus this seq. must be converges. By theorem (5.4.3.b)

Therefore, $(\mathbb{R}, d)$ is complete.

* Example of not complete metric space.

The metric space $(\mathbb{Q}, d)$ is not complete.

Consider, the sequence $\{r_n\}$ such that

$\{r_n\}$ represented in decimal system

$r_n = 1.a_1a_2 \cdots a_n$ is the largest rational number

that satisfy $r_n^2 < 2$

Thus, we obtain the sequence

$r_1 = 1.4$
$r_2 = 1.41$
$r_3 = 1.414$
$r_4 = 1.4142$
$r_5 = 1.41421 \ldots$

For $n > m$

$$d(r_m, r_n) = r_n - r_m = 0.0 \ldots 0\, a_{m+1} \cdots a_n < 1/10^m$$

$\Rightarrow d(r_m, r_n) \longrightarrow 0$ as $m \to \infty$

note that $d(r_1, r_2) = |r_1 - r_2|$, $r_1, r_2 \in \mathbb{Q}$

$\Rightarrow \{r_n\}$ is Cauchy seq.

However, $\{r_n\} \longrightarrow \sqrt{2} \notin \mathbb{Q}$

Thus, the space of rational numbers is not complete.

Problem 3: (30 points)

Let X be a metric space with a dense subset $A \subset X$ such that every Cauchy sequence in A converges in X. Prove that X is complete.

- Let (x_n) be a Cauchy sequence in X

Since A dense in X

we can choose a sequence (a_n) in A such that

$d(x_n, a_n) \longrightarrow 0$ as $n \longrightarrow \infty$

(i.e choose $a_n \in A$ s.t $d(x_n, a_n) < 1/n$)

- Given any $\varepsilon > 0$, $\exists\, M \in \mathbb{N}$ s.t

$d(x_n, a_n) < \varepsilon/3$ and $d(x_m, x_n) < \varepsilon/3$ $\forall\, m, n > M$ so

$$d(a_m, a_n) \leq d(a_m, x_m) + d(x_m, x_n) + d(x_n, a_n)$$
$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

$\Rightarrow (a_n)$ is Cauchy sequence in A

$\Rightarrow (a_n) \longrightarrow x$, for some $x \in X$

- Given any $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t

$d(x_n, a_n) < \varepsilon/2$ and $d(a_n, x) < \varepsilon/2$ $\forall\, n > N$

$$d(x_n, x) \leq d(x_n, a_n) + d(a_n, x)$$
$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

which shows that (x_n) converges to x

Therefore, X is complete.

Problem 4: (30 points)

Let (X, d) be a complete metric space. Let $\{f_n : X \to X, n = 1, 2, \cdots\}$ be a sequence of contractions with the same constant k and let $\{f : X \to X\}$ be a contraction with the same constant k such that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in X$. If the fixed point of f is x^* and the fixed point of f_n is x_n^* for $n = 1, 2, \cdots$, then show that

$$x^* = \lim_{n\to\infty} x_n^*.$$

① $f_n(x_n^*) = x_n^*$ $\quad \forall n = 1, 2, \cdots$

② $f(x^*) = x^*$

if $f_n \longrightarrow f$, let $\varepsilon > 0$ and choose N s.t

Now, $\forall n \geq N$

$$d(x_n^*, x^*) \leq d(f_n(x_n^*), f(x^*))$$

$$\leq d(f_n(x_n^*), f(x_n^*)) + d(f(x_n^*), f(x^*))$$

$$\leq \varepsilon(1+k) + k\, d(x_n^*, x^*)$$

$$(1-k)\, d(x_n^*, x^*) \leq \varepsilon(1-k)$$

$$d(x_n^*, x^*) \leq \varepsilon \qquad \forall n \geq N$$