King Fahd University of Petroleum and Minerals
Department of Mathematics

Math 550 Linear Algebra (Term 211)

Major Exam 1 (Duration = 3 hours)

Problem 1. Let T be the linear operator on $\mathbb{R}^3$ defined by $T(x,y,z) = (y+z, x+z, x+y)$. Let $S := \{e_1, e_2, e_3\}$ be the standard ordered basis for $\mathbb{R}^3$ and let $B := \{f_1, f_2, f_3\}$ with $f_1 = (0,1,-1)$, $f_2 = (-2,0,1)$, $f_3 = (1,-1,0)$.

(1) Find the matrix of T in S.

(2) Show that T is invertible and find T^{-1}.

(3) Show that B is a basis.

(4) Find the matrix of T in B.

Problem 2. Let V be the space of $n \times n$ matrices over a field F and $B := \{A_{ij}\}_{1 \le i,j \le n}$ its standard basis (i.e., A_{ij} takes **1** in the **i***th* row and **j***th* column and **0** elsewhere). Consider the following two subspaces of V:

$$W_1 := \{A \in V \mid \mathrm{trace}(A) = 0\}$$

$$W_2 := \{A \in V \mid \exists\, A_1, A_2 \in V \ \ s.t. \ \ A = A_1A_2 - A_2A_1\}$$

(1) For each $i, j, k, h \in \{1, \ldots, n\}$, find $A_{ij}A_{kh}$

(2) Show that $A_{ij} \in W_2$, $\forall\, i \neq j \in \{1, \ldots, n\}$

(3) Construct a subset $S := \{E_{ij}\}_{i,j}$ of W_2 made of $n^2 - 1$ linearly independent elements.

(4) Show $W_1 = W_2$

Problem 3. Given a matrix A, let f and p denote its (monic) characteristic polynomial and minimal polynomial, respectively. Find f and p for each one of the following matrices:

(1) $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ over $\mathbb{R}$

(2) $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ over $\mathbb{R}$

(3) $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ over $\mathbb{C}$.

Problem 4. Let V be the space of $n \times n$ matrices over a field F and $B := \{A_{ij}\}_{1\leq\ i,j\ \leq n}$ its standard basis (i.e., A_{ij} takes **1** in the **i***th* row and **j***th* column and **0** elsewhere).

Let $A = \begin{pmatrix} & | & \\ - & a_{ij} & - \\ & | & \end{pmatrix}$ be a fixed $n \times n$ matrix over F and T the linear operator on V defined by

$$\begin{array}{rccc} T: & V & \longrightarrow & V \\ & X & \mapsto & AX - XA \end{array}$$

(1) For each $i, j \in \{1, \ldots, n\}$, find $c_1, \ldots, c_n$ and $d_1, \ldots, d_n$ such that

$$T(A_{ij}) = \sum_{k=1}^{n} \left(c_k A_{kj} - d_k A_{ik}\right)$$

(2) Assume A is diagonal with entries $a_1, \ldots, a_n$. Show that T is diagonalizable.

(3) Assume $n \geq 4$ and let $a_1 = a_2 = a_3 = 1$ and $a_i = 0\ \forall\ i \geq 4$. Find the minimal and characteristic polynomials of T.

Problem 5. Let V be the space of $n \times n$ matrices over a field F and $B := \{A_{ij}\}_{1\leq\ i,j\ \leq n}$ its standard basis (i.e., A_{ij} takes **1** in the **i***th* row and **j***th* column and **0** elsewhere).

Let $A = \begin{pmatrix} & | & \\ - & a_{ij} & - \\ & | & \end{pmatrix}$ be a fixed $n \times n$ matrix over F and T the linear operator on V defined by

$$\begin{array}{rccc} T: & V & \longrightarrow & V \\ & X & \mapsto & AX \end{array}$$

Assume that A is diagonalizable; that is, $AP = PD$ for some invertible matrix P and diagonal matrix D with entries $d_1, \ldots, d_n$.

(1) For each $i, j \in \{1, \ldots, n\}$, find DA_{ij}

(2) Find a basis $B' := \{A'_{ij}\}_{1\leq\ i,j\ \leq n}$ for V such that each A'_{ij} is a characteristic vector of T.

(3) Assume $n \geq 4$ and let $d_1 = d_2 = d_3 = 1$ and $d_i = 0\ \forall\ i \geq 4$. Find the minimal and characteristic polynomials of T.

Problem 6. Let V be a finite-dimensional vector space over F, T a linear operator on V, and $W_1, \ldots, W_k$ subspaces of V *invariant* under T such that $V = \bigoplus_{i=1}^{k} W_i$. Let p_o denote the minimal polynomial of T and let p_i denote the minimal polynomial of T_{W_i} for each $i = 1, \ldots, k$.

Assume $p_o = \prod_{i=1}^{k} q_i^{n_i}$ and, for each i, $p_i = q_i^{n_i}$, where $q_1, \ldots, q_k$ are distinct monic irreducible polynomials in $F[x]$ and $n_1, \ldots, n_k$ are nonzero positive integers.

Prove there is a vector α in V such that its T-annihilator p_α is equal to p_o.