King Fahd University of Petroleum and Minerals Department of Mathematics

Math 550 Linear Algebra (Term 211)

Major Exam 2 (Duration = 3 hours)

Problem 1. For each one of the following matrices over **R**, determine its invariant factors and rational form:

$$(1) \quad A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(2) \quad B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$(3) \quad C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Problem 2. Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

- (1) Reduce xI A to its Smith normal form.
- (2) Use the Smith normal form of *A* to find its invariant factors.
- (3) Find the Jordan form of *A*.
- (4) Let *T* be a linear operator on \mathbb{R}^4 such that *A* is the matrix associated to *T* in the standard basis $\{e_1, e_2, e_3, e_4\}$. Find an explicit cyclic decomposition of \mathbb{R}^4 under *T*; namely, find α_1 , $\alpha_2 \in \mathbb{R}^4$ and their respective *T*-annihilators p_1 , p_2 such that $\mathbb{R}^4 = Z(\alpha_1, T) \oplus Z(\alpha_2, T)$.

Problem 3. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 2 \end{pmatrix}$ with $x, y, z \in \mathbb{R}$ and let J denote the Jordan form of A. (1) Assume a = 0. Find J.

(2) Assume $a \neq 0$. Find *J*.

Problem 4. Give all possible 7×7 complex matrices *A* in *rational form* with minimal polynomial $(x+1)^2(x-2)$ and with *four* invariant factors.

Problem 5. Let $A = \begin{pmatrix} 0 & 0 & a \\ 0 & 1 & b \\ a & 0 & 0 \end{pmatrix}$ where *a* and *b* are *nonzero* real numbers.

- (1) Reduce xI A to its Smith normal form.
- (2) Use the Smith normal form of *A* to find its minimal polynomial p_o , and find all values of *a* and *b* for which *A* is diagonalizable.
- (3) Let *T* be a linear operator on \mathbb{R}^3 such that *A* is the matrix associated to *T* in the standard basis $\{e_1, e_2, e_3\}$. Find the respective *T*-annihilators of e_1, e_2 , and e_3 .
- (4) Show that *T* has a cyclic vector; namely, find $\alpha \in \mathbb{R}^3$ such that $\mathbb{R}^3 = Z(\alpha, T)$, and give the matrix of *T* in the basis $S := \{\alpha, T\alpha, T^2\alpha\}$

Problem 6. Let *V* be a finite-dimensional vector space over a field *F* and *T* a linear operator on *V*. Let p_o denote the minimal polynomial of *T* and $V = W_1 \oplus \cdots \oplus W_k$ its primary decomposition. Prove the following:

- (1) If *W* is an *invariant* subspace, then $W = (W \cap W_1) \oplus \cdots \oplus (W \cap W_k)$. [Hint: Use projections]
- (2) If p_0 is *irreducible*, then every *invariant* subspace is *T*-admissible.
- (3) Every *invariant* subspace is *T*-admissible $\iff p_o = p_1 p_2 \cdots p_k$ (i.e., $r_i = 1$, $\forall i$).
- (4) Suppose $F = \mathbb{C}$. Deduce from (3) that: *T* is diagonalizable \Leftrightarrow Every *invariant* subspace is *T*-admissible.